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- Over the last three lectures we learnt the basics of the "Mathematics of Sharing". In all 3 fair-division methods we covered, the key assumption was that the assets S were continuous. However, this is not always the case.

  I recommend the interested students to go over the Section 3.5 of your Textbook which explains one of the most common division methods used for the discrete assets, known under the name "The Method of Sealed Bids". Meanwhile, we will skip this section at the lectures and there will be no problems related to it in the tests.

- To summarize, we learned:
  * the basic concepts of the fair-division games:
    - the assets
    - the players
    - the value system (of each player)
    - fair share
    - fair division
    - fair-division methods
  * 3 Basic fair-division methods for continuous assets:
    - the divider-chooser method (only 2 players) (Sect 3.2)
    - the lone-divider method (≥ 3 players) (Sect 3.3)
    - the lone-chooser method (≥ 3 players) (Sect 3.4)

- Ask if there are any q-s on this topic? (visit OH for more details)

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- This week we will be covering a completely different topic:

"The Mathematics of Getting Around"

Let me first explain what kind of problems we are interested in and provide the key examples from the everyday life.

Routing problems are concerned with finding ways to route the delivery of services/goods (e.g. mail, packages, garbage collection, meter reading) to an assortment of the destinations (homes, companies, warehouses etc.)

The two Key Questions in such Problems are:

- Is an actual route (satisfying certain conditions) is possible? (this will be referred to as the existence question)
- If the answer is Yes (to the above q-n), then the next q-n is to find the most "optimal" route (the optimization question). Here, the "most optimal" may refer to the least expensive, or the fastest one, or the shortest one.

This week we will address only a particular case of the routing problems, called the street-routing problems. The key feature will be the requirement for the route to pass through specified connections (i.e. each street / each bridge / ... must be covered by the route).

The most common examples with this requirement are: mail delivery, snow removal, garbage collection, tour buses, etc. Before we switch to the mathematical model of such problems, let us provide several examples.

Example 1: The Mail Carrier Problem.

In the neighborhood of Stony Brook Village, a mail carrier has to deliver mail to all the houses. The mail carrier must make two passes through blocks with buildings on both sides of the street and just one pass through blocks with buildings on only one side of the street. Moreover, we assume that the mail carrier must start and end his route at the local post office.

Task: Find the most optimal route that would allow the mail carrier to cover all the neighborhood with the least amount of walking.

Example 2: The UPS driver problem.

Now we consider a similar problem, where the UPS driver has to deliver packages in the Stony Brook Village (he has a map of the neighborhood with locations of destinations marked by RED color: he doesn't bring packages to every house). Therefore, the UPS driver must pass only through the streets containing the marked houses and we assume that he can bring packages to both sides of the street, when driving one way. So we request the UPS driver to pass these streets just once. Finally, there is no requirement to start and end at the local post office! Instead, he will be assigned a street which he will enter in the beginning, and also a street he will have to leave through.

Task: Find the shortest route satisfying these requirements.

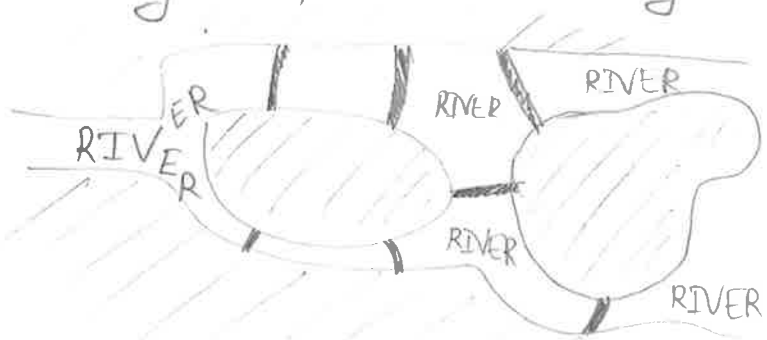
Example 3: The Security guard Problem.

Now we consider a similar problem, but with a security guard or policeman patrolling the streets of the Stony Brook Village from 2a.m. to 3a.m. The task for a security guard is to pass through each street once. He starts and ends at the harbor, where he parks his car.

Task: - If possible find the route which passes through each street exactly once, starting and ending at the harbor.
- If the former is not possible, then what is the shortest route which still covers each street at least once and starts/ends at the harbor?

Example 4 The Königsberg Bridges Puzzle.

The city map looks as follows:



The two islands and the north & south banks are connected by 7 bridges indicated above by

Task: - If possible find a route to walk around the city so that each bridge is crossed exactly <u>once</u>
- If the former is not possible, then what is the smallest number of times you need to cross bridges so that each bridge is crossed at least once?

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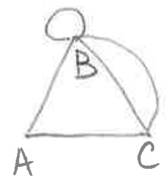
To answer those questions, we need to introduce some new mathematical machinery, invented by Euler in ~1700.

- **Graphs** (Section 5.2 of your textbook).

Graph is an ordered pair (V, E), where V denotes the set of vertices, while E denotes the set of edges.

The simplest way to think about graphs is by drawing the corresponding picture

Ex 1:



refers to the graph with the set of vertices $V = \{A, B, C\}$ and the set of edges $E = \{AB, BC, BC, CA, BB\}$.

! The order is irrelevant, e.g. $AB = BA$

Note: - We listed BC twice as there are two edges b/w B & C

- We listed BB which stays for an edge starting and ending at B.

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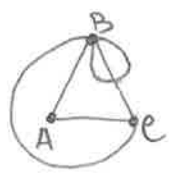

Though this picture is completely different from the above one, it corresponds to the same graph.

Definitions:

- two vertices are adjacent if they are connected by an edge.
- an edge connecting a vertex with itself is called a loop.
- the degree of a vertex is the number of edges meeting at that vertex (above: $\deg(A) = 2$, $\deg(B) = 5$, $\deg(C) = 3$).
- a vertex is called even (respectively, odd) if its degree is an even number (respectively, odd number)

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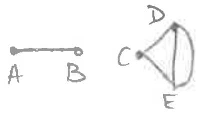
Definitions:

( etc.)

- a clique is a graph where any two vertices are connected by exactly 1 edge.
- graph is called a simple graph if there are no loops and multiple edges.
- the graph like  is called disconnected (no way to get from B to C)

The connected pieces of any disconnected graph are called the components

Example: If a small bus company runs service between 6 villages, then we can efficiently represent that by a graph on 6 vertices with an edge connecting two vertices if there is a direct bus between these 2 cities.

(a) What does it mean that this graph is connected?

(b) What does it mean that this graph is a clique?

Definitions:

- two edges of a graph are called adjacent if they have a common vertex (consider example of )
- a path is a sequence of distinct edges, each adjacent to the next one.
- the length of the path is the number of edges in this sequence.

(e.g. above ABCEDC is a path, but BCDEBC is not a path) b/c BC appears twice!

- a circuit is a path in which "end-point" is the same as the "starting point" (i.e. it is a closed path) For example, any loop is a circuit of length 1
- an Euler path is a path that covers all edges of the graph
- an Euler circuit is a circuit that covers all edges of the graph.