Lecture #11 09/23/2016

• Last time, we learned the basics of the graph theory.
Of particular interest for today:
- the degree of each vertex
(which leads to the notion of even / odd vertices)
- the path and circuit
- the Euler path and Euler circuit

Rmk: Any loop contributes 2 to the degree of the corresponding vertex

* We call a graph without loops and multiple edges a simple graph.
* In the circuit it is irrelevant where is a starting (= ending) point

The key notion for today's discussion is the notion of "bridges".
A bridge in a connected graph is an edge such that erasing it, we obtain a disconnected graph.

• Euler Theorems
The key questions we have are:
Q1: Given a graph, determine if it admits an Euler path, an Euler circuit, or none of them?

Q2: If there exists an Euler path / circuit, how do we find it?

Let us first address the first question. This goes back to Leonardo Euler (~1700), who proved the following simple, but remarkable theorems

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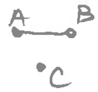
* **Thm 1** (Euler's Circuit Theorem): A graph has an Euler circuit if and only if
  (a) It is a connected graph.
  (b) All vertices are <u>even</u>, i.e. there are <u>no</u> vertices with odd degree.

<u>Rmk</u>: There is a subtle q-n of what to do in case we have isolated vertices, e.g. 

. We will agree that we still want to travel along all edges once, visiting each vertex, so that there is no Euler path / circuit in this example.
[The book does not address this issue and we have to agree from the beginning on how to deal with isolated points

* <u>Thm 2</u> (Euler's Path Theorem): A graph has an Euler path if and only if
  (a) It is a connected graph.
  (b) There are <u>(exactly) zero or two odd vertices</u>, the rest being <u>even</u> vertices.

Any Euler path <u>must</u> start at one of those odd vertices and end at the other odd vertex.

* We also have a bit unrelated result:

<u>Thm 3</u> (Euler's Sum of Degrees): The sum of the degrees over all vertices of a graph equals twice the number of edges in this graph.

► Any edge (or ) contributes 2 to the total sum of degrees, hence, the result ∎

<u>Cor</u>: Any graph has even number of <u>odd</u> vertices.

► A sum of integer numbers is even if and only if the number of odd numbers is even ∎

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Upshot (of the previous three theorems)

- If the #{odd vertices} = 0 ⇒ there exists an Euler circuit
- If the #{odd vertices} = 2 ⇒ there exists an Euler path, but not an Euler circuit
- If the #{odd vertices} = 4, 6, 8, ... ⇒ there is neither an Euler path nor an Euler circuit.
- The situation with #{odd vertices} = 1, 3, 5, 7, --- – not possible at all!

But if the #{odd vertices} is 0 or 2, the above Theorems don't tell us how actually to find such an Euler circuit or path.

- Fleury's Algorithm

Given a graph with no odd vertices or with exactly two odd vertices, Thms 1, 2 predict the existence of an Euler circuit and Euler path, respectively. There might be several of those, but our key task is to construct at least one of possible Euler circuits/paths

This can be achieved by the Fleury's algorithm. We will specify a step-by-step algorithm with several possible choices at each step, which will eventually provide an Euler's circuit or path after a finite number of steps.

* Step 0: Choose a starting point (vertex):
  - If there are no odd vertices, pick any vertex
  - If there are exactly two odd vertices, pick either of them.

* Step 1: Choose any edge adjacent to the starting vertex, which is not a bridge. "Walk" along it and temporarily "erase/forget" this edge.

* Step 2, 3, ... : Do as in Step 1 until you can't make any more "walks"

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Performing all previous steps we will end up at a step, when we cannot find any edges adjacent to the current vertex, which are not the bridges for the remaining part of the graph.

! It turns out that at this moment, we have accomplished our goal: we found an Euler path or circuit!

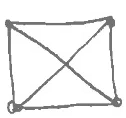
Rmk: For your convenience, it is recommended to keep two copies of your graph (drawn in pencil). As you proceed with the above algorithm, you erase the edges you already "walked" on one copy of your graph, while you put the number of the corresponding step and the direction's arrow over this edge in another copy of the graph.

At the end of the algorithm, first copy will be completely erased, while the second copy will show the Euler path / circuit.

- Examples

Let us illustrate this by a couple of examples.

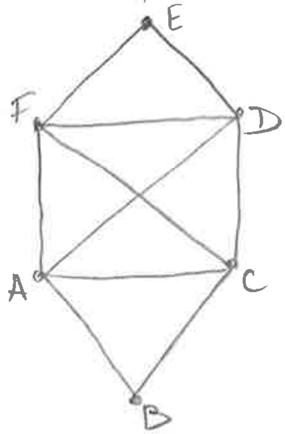
Ex 1: Are there Euler circuits or Euler paths in the graph

 – clique on 4 vertices. If so, construct one?

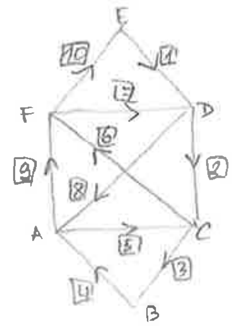
➤ The degree of each of the four vertices is 3.

Hence $\#\{\text{odd vertices}\} = 4 \overset{\text{Thms 1,2}}{\Longrightarrow}$ there are no Euler paths nor Euler circuits ∎

Ex 2: Does the graph from below admit an Euler circuit or an Euler path? If yes, provide one of such.



One of the possible Euler circuits:



- Clearly $\deg(A) = \deg(C) = \deg(D) = \deg(F) = 4$
  $\deg(B) = \deg(E) = 2$.

So there are no odd vertices $\xRightarrow{\text{Thm 1}}$ there exists an Euler circuit.

To construct it, we apply the Fleury's algorithm.

* Pick any vertex at Step 0, e.g. E.
* Go to D or F (neither EF nor ED is a bridge), e.g. go to D ⇝ erase ED
* Go to F, A, or C (none of DF, DA, DC is a bridge), e.g. go to C ⇝ erase DC
* Go to F, A, or B (none of CF, CA, CB is a bridge), e.g. go to B ⇝ erase CB
* Only 1 option: going to A ⇝ erase BA
* Go to F, D, or C (no bridges), e.g. to C ⇝ erase AC
* Only 1 option: going to F ⇝ erase CF
* Out of 3 options, only 1 works: go to D ⇝ erase FD
* Only 1 option: going to A ⇝ erase DA
* Only 1 option: going to F ⇝ erase AF
* Only 1 option: going to E ⇝ erase FE. THE END