Lecture #14

10/03/2016

- Today we are gonna start a new topic "Mathematics of Touring". The problems of that sort are usually known as "traveling salesman problem" (TSP).

Warning: The name has been used as a metaphor and nowadays rarely refers to the real salesman

The key elements are:

- A traveler (e.g. individual, group, bus, bee, etc.)
- A set of sites (places to visit)

  N = Number of sites
- A set of costs (expense of traveling from one site to another, could be represented by distance, time, \$ expenses)

  positive numbers

Def: A solution to a TSP is a trip starting and ending at a site, and visiting all other sites exactly once. This trip is called a tour. An optimal solution is a tour with minimal total cost.

Discuss (and illustrate) the following examples from the Textbook:

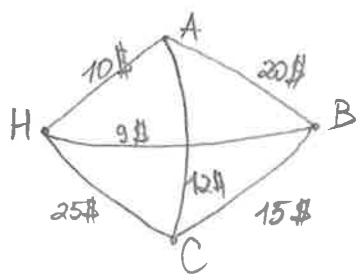
- Example 6.1 (Salesman's trip)
- Example 6.2 (Interplanetary Mission)
- Example 6.3 (Concert Tour)
- * School buses

Read at home!

To summarize, the key distinction from our previous topic is that any two sites are connected by edge (so we have a clique with vertices representing sites), but every edge "has its own cost", i.e. we look for the shortest route, where all edges are of different length.

The basic example is as follows:

Ex 1: Bob is a salesman. His neighbourhood consists of 4 villages (A, B, C, H). (H: Home)
Bob lives in H and wants to travel around, starting in his village, visiting any other (exactly once!) and returning to his village. For any 2 villages, there is a direct bus b/w them with the cost depicted in the following chart:



Question: What is the cheapest tour he should choose?

We will find an answer to this question next time, but to give you a flavor, note that e.g. the total cost of the routes

H − A − B − C − H is $10 + 20 + 15 + 25 = 70$ \$

H − B − A − C − H is $9 + 20 + 12 + 25 = 66$ \$.

You may ask if those are all the options or not?
We will address this in a moment.

- As in the previous topic, the mathematical model of our real-life problem will be represented by a graph, while the notion of a route leads us to define Hamilton path and Hamilton circuit

N.B.: This looks similar to Euler path/circuit, but essentially differs from the latter.

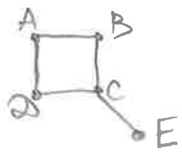
- <u>Hamilton path</u> (in a connected graph) is a path that visits all the vertices of the graph exactly once.

- <u>Hamilton circuit</u> (in a connected graph) is a circuit that visits all the vertices of the graph exactly once.

<u>Example 2</u>:

(a)  has a Hamilton circuit, e.g. 

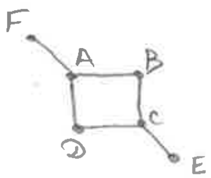
a Hamilton path which is not a circuit 

(b) 

has a Hamilton path, e.g. 

but <u>does not</u> have a Hamilton circuit

(if there was one, then getting into E, you are stuck there)

(c) 

has neither a Hamilton path nor a Hamilton circuit

(by the same reason as in (b), E or F is the starting point, while the ending pt must be the other of {E, F}. However, it is clear that in that case you can't visit A, B, C, D exactly once)

<u>Upshot</u> (of that example):

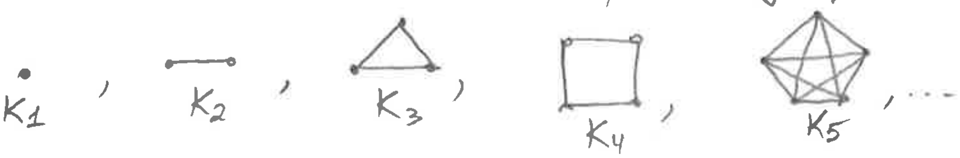
- some graphs have neither Hamilton paths nor Hamilton circuits
- some graphs have Hamilton paths, but do not have Hamilton circuits
- ! if a graph has a Hamilton circuit, then it also has many Hamilton paths, as you can delete one of the edge in the circuit.

<u>Warning</u>: However, not all Hamilton paths come that way (i.e. by deleting an edge in Hamilton circuit).

In the case of TSP, we will always have a complete graph

A complete graph is a graph without loops and with exactly 1 edge between any pair of different vertices.

We will denote a complete graph with N vertices by K_N.



Note that: - the degree of every vertice is $N-1$.
- the number of edges in K_N is $\frac{N(N-1)}{2}$
(this follows from $\#\{\text{edges}\} = \frac{1}{2}\cdot(\text{sum of degrees of vertices})$)

Question: (1) What is the number of Hamilton circuits in K_N?
(2) What is the number of Hamilton paths in K_N?

Observation #1: Given a circuit, we can actually start equally well from any vertex in it, and follow arrows.

In view of this observation, we will assume that we are given the starting point (= ending point) and want to count all circuits in K_N starting and ending at that vertex.

Answer 1: The number of Hamilton circuits in K_N equals
$1\cdot 2\cdot 3\cdot \ldots \cdot (N-2)\cdot (N-1) =: (N-1)!$ (this is a common notation "!" is spelled as "factorial")

- Starting at a given vertex, on the first step we have $(N-1)$ options on where to go. On the second step we have $(N-2)$ options, etc.
In total, we get $(N-1)\cdot(N-2)\cdot \_\_ \cdot 2\cdot 1$

We see that there are $2! = 2$ Hamilton circuits in K_3, $3! = 1\cdot 2\cdot 3 = 6$ - in K_4, $4! = 1\cdot 2\cdot 3\cdot 4 = 24$ - in K_5, $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$ - in K_6, ...