Lecture #17 10/10/2016

· Last week: Traveling Salesman Problems.
We learnt 4 algorithms:
- The Brute-Force Algorithm
- The Nearest-Neighbor Algorithm
- The Repetitive Nearest-Neighbor Algorithm
- The Cheapest-Link Algorithm

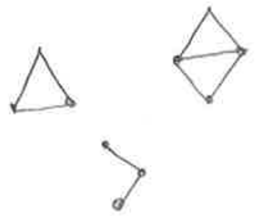
} approximate algorithms

◦ Ask if there are any q-s on this material and on hwk.
Rmk: Spell out again what breaking a circuit means.

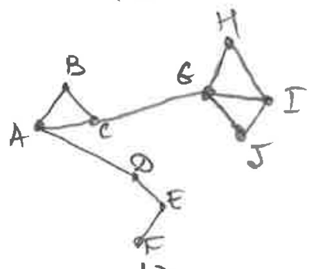
• Today: The Mathematics of Networks [Section 7 in your Textbook]
◦ Give examples of Networks from real life as an introduction.

A network is a fancy name for a connected graph
(in that context, we will often call vertices as nodes, edges as links)
Rmk: In most of our examples, we will deal with simple networks, i.e without loops and multiple edges, but this is not a general requirement.

Example 1: In a class with 10 students, some of them exchanged facebook info and became friends. In the example (a), we have a disconnected graph ⇒ it is not a network.


(a)

But: in the example (b), the graph is connected and, hence, we have a network.



(b)

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Since a network is a connected graph, there is always a path between any two vertices.

<u>Def</u>: The <u>degree of separation</u> of any 2 vertices is the length of the shortest path between them.

<u>Example 2</u>: In the setting of Example 1(b), the degree of separation of B and J is 3, of F and C is 4, of C and G is 1

The following observation characterizes complete graphs:

<u>Rem</u>: In a simple graph (i.e. no loops and no multiple edges), all the degrees of separation are 1 <u>iff</u> this is a complete graph

! The following examples mimicks on a small scale very important q-s in real life: making telecommunications network, constructing power grid, creating underground fiber-optic lines, etc. between cities, so that any two cities are connected by a path, while the total cost is the minimal possible.

<u>Example 3</u>: In a nbhd with 6 villages, there are 7 roads b/w some of the villages, making them into a network. (see picture)

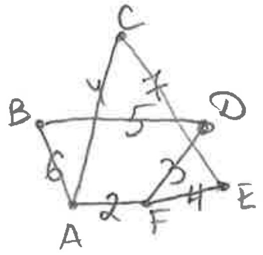

Fig 1

The numbers represent the cost of laying a new power line between those 2 cities (the ends of the edge).

<u>Goal</u>: Find the cheapest way to construct power lines along some of the roads, so that any two villages are connected by a path.

<u>Terminology</u>: A network with numbers over all of its edges is called a <u>weighted network</u>

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In the example 3 above, one of the options is to erase edges CE and BD, so that we construct lines b/w the following pairs of villages:

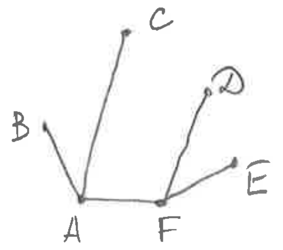

Fig. 2

Note: We cannot discard any remaining edge, since this will break the graph on the left into several connected components.

To summarize, the goal of Example 3 is to find a spanning connected subgraph of the minimal cost.

* given a graph with the set of vertices and edges, a subgraph is another graph, whose ~~sets~~ of vertices and edges are sub~~sets~~ of the corresponding sets in the ambient graph.

* spanning refers to the setting when the subgraph has the same set of vertices as the ambient graph.

Q: What can we say about these cheapest spanning connected subgraphs?

Let's look at the discussion above (Fig. 2). The subgraph over there doesnot have circuits and has 5 edges.

Observation 1: Assuming that all the weights (numbers on edges) are positive, it is clear that if the subgraph has a circuit, we can delete one of its edges, so that the remaining subgraph is still connected, but its total cost is less.

Thus: we want to look for subgraphs without circuits

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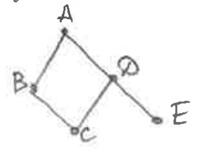
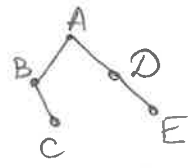
The above Observation 1 brings us to the following definition:

! Def: A tree is a network (i.e. connected graph) without circuits

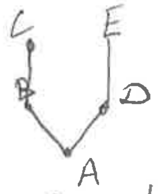
Example 4:

(a) 

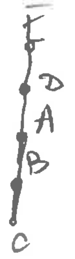
not a tree: there is a circuit ABCDA

(b) 

is a tree

(c) 

is a tree

(d) 

is a tree

$\underbrace{\text{(b), (c), (d)}}$ these are the same trees, i.e. they have the same vertices and edges, even though the pictures look a bit different.

Terminology: The tree is rooted at A in the pictures (b,c)
The tree is rooted at C in the picture (d)

All trees satisfying the following 3 properties:

① For any 2 vertices, there is a single path connecting them

!② All edges are bridges

!③ The number of edges is exactly 1 less the number of vertices (i.e. if # vertices $= N$, then # edges $= N-1$).

Corollary 1 (of ③): In a (connected graph =) network with M edges, N vertices, we always have $M \geq N-1$. If $M = N-1$, it is a tree. If $M > N-1$, there must be circuits.

Def: The number $M-(N-1)$ is called the redundancy of the network.

Corollary 2 (of ②): If you want to minimize number of edges or the total weight (assuming all weights are positive) of a connected spanning subgraph, this subgraph must be a tree!

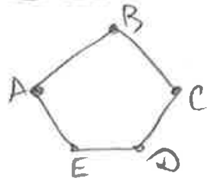
Combining discussions on pages 3 and 4, we get to the notions:

- subtree of a graph — a subgraph which is a tree
- a spanning (sub)tree — a subtree which includes all vertices of the ambient graph.

To address the goal of Example 3, we thus have to pick the optimal of all spanning subtrees. While the question of optimizing will be addressed next time, let us conclude our ~~today's~~ discussion by "counting spanning (sub)trees".

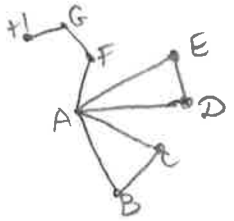
Example 5:

(a)



Breaking the circuit by deleting one of its edges makes ~~it~~ into a tree. Hence, number of spanning trees in case (a) is $\underline{5}$.

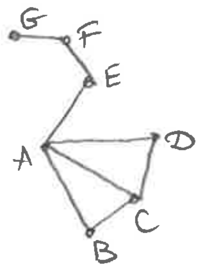
(b)



The edges HG, GF, FA are bridges ⇒ can't delete them. The other edges split into 2 circuits: AE, ED, DA
AC, CB, BA.

We must break each of them, i.e. delete one of the edges in each list of 3. After that we have only 7 edges, while #vertices = 8 ⇒ automatically see it is a tree. Hence there are $3 \cdot 3 = \underline{\underline{9}}$ different spanning trees.

(c)



As before GF, FE, EA – Bridges ⇒ can't delete them. In contrast to (b), the two circuits ADC and ACB have common edge AC.

Case 1: If AC is deleted, then we are left with 1 circuit formed by edges AD, DC, CB, BA ⇒ delete 1 of them. So, we have $\underline{4}$ choices.

Case 2: If AC is not deleted, then we must delete AD or DC in 1st circuit and CB or BA in 2nd circuit ⇒ $2 \cdot 2 = \underline{\underline{4}}$ choices

Total: $4+4 = \boxed{8}$