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- To clarify some q-s from last time, let us recall that:

  * In an arithmetic sequence or in a population sequence of the population with linear growth model, we have:

$$\boxed{P_0 + \dots + P_{n-1} = \frac{P_0 + P_{n-1}}{2} \cdot n}$$

  or more generally:

$$\boxed{P_a + P_{a+1} + \_\_ + P_b = \frac{P_a + P_b}{2} \cdot (b-a+1)}$$

  * The growth rate is given by

$$r = \frac{Y-X}{X} \quad \text{, equivalently: } r = \frac{Y-X}{X} \cdot 100\%$$

<u>Note</u>: • Given X and r, we can determine Y as follows:

$$\frac{Y-X}{X} = r \Rightarrow Y - X = r \cdot X \Rightarrow \boxed{Y = (1+r) \cdot X}$$

• Given Y and r, we can determine X as follows:

$$\frac{Y-X}{X} = r \Rightarrow Y = (1+r)X \Rightarrow \boxed{X = \frac{Y}{1+r}}$$

- Now let us switch back to the discussion of our second population model.

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Def: A population growth exponentially if in each generation the population growth by the same constant factor R, called the common ratio.

In this case, the population sequence looks as follows:

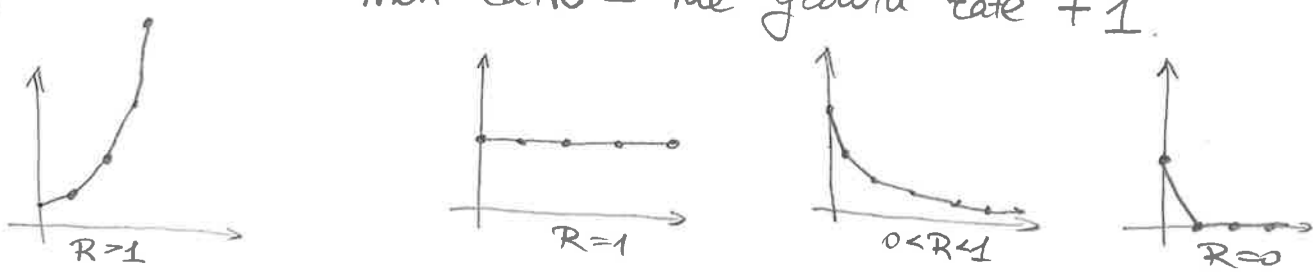
$$P_0,\ P_1 = R\cdot P_0\ ,\ P_2 = R^2 P_0,\ P_3 = R^3 P_0, \_\_\_$$

* Explicit f-la: $P_N = R^N \cdot P_0$
* Recursive f-la: $P_{N+1} = R\cdot P_N$
* $R \geq 0$ as all terms must be non-negative

Note: The population sequence from above is a geometric sequence.

! $\boxed{R = 1 + r}$ i.e. the common ratio = the growth rate + 1.

Visualizing



! The exponential growth models are used to describe the population growth under unrestricted breeding conditions.

• Geometric Sum Formula

Similarly to the case of arithmetic sum, we want to compute the sum of the few first terms of geometric sequence.

$$\left.\begin{aligned} S &= P_0 + P_1 + P_2 + \_\_\_ + P_{n-1} \\ R\cdot S &= \quad\ R\cdot P_0 + R\cdot P_1 + \_\_\_ + R\cdot P_{n-2} + R\cdot P_{n-1} \end{aligned}\right\} \Rightarrow (R-1)\,S\ \underbrace{R\cdot P_{n-1}}_{P_n} - P_0$$

$$\Rightarrow \boxed{S = \frac{P_n - P_0}{R-1}.} \quad \text{or} \quad \boxed{P_0 + R P_0 + \_ + R^{n-1} P_0 = \frac{(R^n - 1)P_0}{R-1}}$$

Discuss Exercises 9.3.37 and 9.3.47 from textbook

(see next page)

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Ex1 (Exercise 9.3.37): A population grows according to an exponential growth model. The initial population is $P_0 = 11$ and the common difference is $R = 1.25$

(a) Find P_1, (b) Give an explicit f-la for P_N, (c) Find P_9.

(a) $P_1 = P_0 \cdot R = 11 \cdot 1.25 = 13.75$

(b) $P_N = P_0 \cdot R^N = 11 \cdot 1.25^N$

(c) $P_9 = 11 \cdot 1.25^9 \approx 82$ ∎

Ex 2 (Exercise 9.3.47) Consider the geometric sequence $P_0 = 2, P_1 = 6, P_2 = 18, \ldots$

(a) Find the common ratio R.

(b) Find $P_0 + P_1 + \_\_ + P_{20}$

(a) $R = \frac{P_1}{P_0} = \frac{6}{2} = 3$

(b) $P_0 + P_1 + \_\_ + P_{20} = P_0 \cdot \frac{R^{21} - 1}{R - 1} = 2 \cdot \frac{3^{21} - 1}{2} = 3^{21} - 1$ ∎

Rmk: The most common example of an exponential growth in real life is a spread of something over community, e.g. news or epidemic

- The third and last population growth model we are going to discuss:

"The Logistic Growth Model"

N.B.: This model is used to describe the growth of biological populations living in a fixed habitat and whose growth rates are in direct proportion to the amount of "elbow room" in the habitat.

Let C be the carrying capacity of a given habitat, and given a population of size P_N, we define

$$\boxed{p_N = \frac{P_N}{C}}$$ – the p-value of the population

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The initial input into the logistic equation is:

- C - the carrying capacity
- r - the growth parameter
- P_0 - initial population $\Longleftrightarrow$ $p_0 = \frac{P_0}{C}$ - the seed.

Logistic Growth Model : $\boxed{p_{N+1} = r \cdot (1-p_N) \cdot p_N}$ i.e. $P_{N+1} = r \cdot (1 - \frac{P_N}{C}) \cdot P_N$

Rmks :
- changing p_0 to $1-p_0$, we get the same population P_N for $N=1,2,3,\ldots$
- Since $0 \le p_N \le 1$, we are forced to require $0 \le r \le 4$.
- Unlike the linear growth and exponential growth models, there is no way to rewrite this recursive f-la as an exact f-la for the population size.

! Consider examples 9.18 - 9.23 from your Textbook at home.

Ex3 (Exercise 9.4.53): A population growth according to the logistic growth model, with the growth parameter $r = 0.8$. Starting with initial population given by $p_0 = 0.3$

(a) Find p_1, (b) Find p_2, (c) determine what percent of the habitat's carrying capacity is taken up by the 3rd generation.

(a) $p_1 = r \cdot (1-p_0) \cdot p_0 = 0.8 \cdot 0.7 \cdot 0.3 = 0.168$

(b) $p_2 = r \cdot (1-p_1) \cdot p_1 = 0.8 \cdot 0.832 \cdot 0.168 \approx 0.1118$

(c) $p_3 = r \cdot (1-p_2) \cdot p_2 \approx 0.07945 \Rightarrow \sim 7.945\%$ is taken up ■

Ex 4 (Exercise 9.4.59): In the logistic growth model $r = 2.8$, $p_0 = 0.15$. Check that the population size stabilizes at $\frac{9}{14} \approx 64.29\%$