Lecture #30

11/11/2016

- Last time: Fibonacci numbers

Ex 1 (Exercise 13.1.8): Given that $F_{32} = 2{,}178{,}309$ and $F_{33} = 3{,}524{,}578$,

(a) find F_{34}

(b) find F_{35}.

(a) $F_{34} = F_{33} + F_{32} = \begin{array}{r} 3524578 \\ +\ 2178309 \\ \hline 5{,}702{,}887 \end{array} = \boxed{5{,}702{,}887}$

(b) $F_{35} = F_{34} + F_{33} = \begin{array}{r} 5702887 \\ +\ 3524578 \\ \hline 9227465 \end{array} = \boxed{9{,}227{,}465}$ ∎

Ex 2 (Exercise 13.1.13(c)): Find the value of "?" such that

$$F_1 + F_3 + F_5 + F_7 + \_\_ + F_N = F_?$$

$\uparrow$ odd number

$N=3 \Rightarrow F_1 + F_3 = 1+2 = 3 = F_4$

$N=5 \Rightarrow F_1 + F_3 + F_5 = F_4 + F_5 = F_6$

$N=7 \Rightarrow F_1 + F_3 + F_5 + F_7 = F_6 + F_7 = F_8$

$\vdots$

$F_1 + F_3 + \dots + F_{N-2} + F_N \overset{\text{by assumption we know for } N-2}{=} F_{N-1} + F_N = F_{N+1}$

So: "?" $= N+1$. ∎

- The "Golden Ratio"

Def: The golden ratio ϕ is defined as $\boxed{\phi := \frac{1+\sqrt{5}}{2}}$

$\uparrow$ Greek letter "phi"

Rmk: ϕ is an irrational number, with decimal approximation

$$\phi \approx 1.6180339887\ldots$$

Golden property: $\boxed{\phi^2 = \phi + 1}$ (*)

As we saw last time, ϕ is a unique positive solution of (*) Explain Again!

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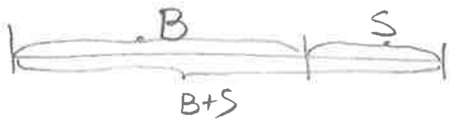
• The Divine Proportion

The golden ratio was first considered by ancient Greeks. They figured out that an ideal proportional split of anything is given by golden ratio.



$$\boxed{\text{Divine Proportion: } \frac{B}{S} = \frac{B+S}{B}}$$

If we denote $\frac{B}{S}$ by x, then we get $x = 1 + \frac{1}{x} \iff x^2 = x+1$.

So x satisfies the golden property and $x > 0 \implies x = \phi. \implies \boxed{\frac{B}{S} = \phi}$

! Starting from ancient Greeks, the golden ratio has been used by painters, architects, designers, etc.

Examples: The ratio of sides of a credit card as well as the ratio of sides of a MacBook is ≈ 1.6 which is almost ϕ.

the difference is really invisible to human eyes.

• Relation between Fibonacci numbers and the Golden Ratio

(i) Binet's Formula from last time can be written as:

$$\boxed{F_N = (\phi^N - (1-\phi)^N)/\sqrt{5}} \quad \text{or} \quad \boxed{F_N = \llbracket \phi^N/\sqrt{5} \rrbracket}$$

(ii) $\phi^2 = \phi + 1 \xRightarrow{\cdot\phi} \phi^3 = \phi^2 + \phi = 2\phi + 1 \xRightarrow{\cdot\phi} \phi^4 = 2\phi^2 + \phi = 3\phi + 2 \xRightarrow{\cdot\phi} \phi^5 = 3\phi^2 + 2\phi = 5\phi + 3 \ldots\ldots$

We see that $\boxed{\phi^N = F_N\phi + F_{N-1}}$

(Just note that $(F_N\phi + F_{N-1}) \cdot \phi = F_N \cdot \phi^2 + F_{N-1}\phi = F_N(\phi+1) + F_{N-1}\phi = \underbrace{(F_N + F_{N-1})}_{F_{N+1}}\phi + F_N$)

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(iii) It turns out that the ratio of two consecutive Fibonacci numbers $\frac{F_{N+1}}{F_N}$ "approaches" the golden ratio ϕ as N becomes bigger and bigger.

N.B.: Mathematically, we say that ϕ is a limit of $\frac{F_{N+1}}{F_N}$ and write $\frac{F_{N+1}}{F_N} \longrightarrow \phi$

Ex3 (Exercise 13.2.33) Consider the quadratic equation

$$F_N \cdot x^2 = F_{N-1} \cdot x + F_{N-2}.$$

(a) Show that $x=1$ is one of the two solutions of the equation.

(b) Without using the quadratic formula, find the second solution.

(a) For this part, we don't need to use standard f-la $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ for the solutions of $ax^2+bx+c=0$.

Instead, let us just plug in $x=1$:

$F_N \cdot x^2 |_{x=1} = F_N$, $\quad (F_{N-1} \cdot x + F_{N-2})|_{x=1} = F_{N-1} + F_{N-2} = F_N$ $\Rightarrow$ indeed a solution as $F_N = F_N$

(b) By aforementioned f-la $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow x_1 + x_2 = -\frac{b}{a}$

In our case: $F_N x^2 = F_{N-1} \cdot x + F_{N-2} \iff F_N \cdot x^2 - F_{N-1} x - F_{N-2} = 0.$

So: $a = F_N$, $b = -F_{N-1}$, $c = -F_{N-2}$ $\Rightarrow -\frac{b}{a} = \frac{F_{N-1}}{F_N}.$

Hence, $x_1 + x_2 = \frac{F_{N-1}}{F_N}$. But according to (a) we have $x_1 = 1 \Rightarrow$

$\Rightarrow x_2 = \frac{F_{N-1}}{F_N} - 1 = \frac{F_{N-1} - F_N}{F_N} = \boxed{\frac{F_{N-2}}{F_N}}$