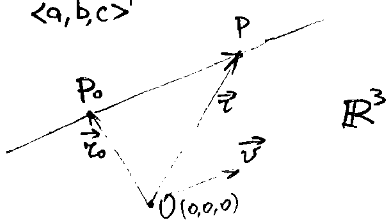
Lecture #3 09/07/2017

- Remind:
  - Office hours: Tu, We $2^{00}-3^{30}$ pm.
    ↳ my office LOM 219-C or Math Dept 442-C
    (if more than 3 students show up)
  - Peer tutors: additional help with the material.

- Equations of Lines and Planes (Sect 12.5)

Lines

A line L in $\mathbb{R}^3$ is determined once we know a point $P_0(x_0, y_0, z_0)$ on L as well as the direction of L. For the latter, we just need a non-zero vector $\vec{v}$ parallel to L.
$\vec{v} = \langle a, b, c \rangle$



$$\left.\begin{array}{l} \overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P} \\ \overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle \\ \overrightarrow{P_0P} = t \cdot \vec{v} = \langle ta, tb, tc \rangle \end{array}\right\} \Rightarrow \overrightarrow{OP} = \langle x_0+ta, y_0+tb, z_0+tc \rangle$$

Upshot: A vector eq-n of L is $\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$

Once a system of coordinates is chosen, we recover parametric eq-n
$\boxed{x = x_0 + at, \ y = y_0 + at, \ z = z_0 + at}$ (t runs through all $\mathbb{R}$)

Rmk: Any line has a lot of different parametric equations due to:
(1) change of a "starting point" P_0 on L
(2) change of the vector $\vec{v}$ (any multiple of it still works)

E.g. $x = t, y = 2t, z = 3t+1$ and $x = 1+3t, y = 2+6t, z = 4+9t$ determine the same line

Ex 1: (a) Find a vector equation and parametric eq-n for the line that passes through $(-3, 5, 1)$ and parallel to the vector $\langle \frac{1}{2}, -\frac{1}{3}, 1 \rangle$.
(b) Find a point on this line whose z-coordinate is ZERO.

Def: The numbers a, b, c (coordinates of $\vec{v}$) are called direction numbers of L.

Finally, from above eq-n we get $(t =) \dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$ (eliminating t).
assuming a, b, c are nonzero

Lecture #3

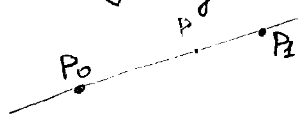
09/07/2017

Following the end of previous page, we get the symmetric eq-n of L:

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

Ex2: Find parametric and symmetric equations of the line that passes through $A(-3,5,1)$ and $B(1,4,-2)$.

Rmk: If we need only a segment P_0P_1, which is the locus of points on the line L passing through P_0, P_1, then its eq-n is similar to that of L. Namely, if $\vec{r_0}, \vec{r_1}$ are vectors $\vec{OP_0}$ and $\vec{OP_1}$ resp., then



$$\vec{r} = \vec{OP} = \vec{OP_0} + \underbrace{t}_{\text{runs from 0 to 1}} \cdot \vec{P_0P_1} = \vec{r_0} + t\cdot(\vec{r_1} - \vec{r_0}) = (1-t)\vec{r_0} + t\cdot\vec{r_1}$$

$$\boxed{\vec{r} = (1-t)\vec{r_0} + t\,\vec{r_1}\,,\ 0 \le t \le 1}$$

Ex3: Determine whether the lines L_1, L_2 are parallel, skew or intersecting:

L_1: $x=-2+t,\ y=1-2t,\ z=3t$

L_2: $x=1-s,\ y=2+3s,\ z=-2s$

Planes

A plane in $\mathbb{R}^3$ is determined by a point $P_0(x_0,y_0,z_0)$ in the plane and a vector $\underbrace{\vec{n}}_{\text{normal vector}}$ that is orthogonal to this plane.

Then a point $P(x,y,z)$ belongs to this plane iff $\overbrace{\vec{P_0P}}^{\vec{OP}-\vec{OP_0}=\vec{r}-\vec{r_0}} \perp \vec{n} \Leftrightarrow \vec{n}\cdot\vec{P_0P}=0$

Thus we obtain a

- vector eq-n of the plane: $\boxed{\vec{n}\cdot(\vec{r}-\vec{r_0})=0 \Leftrightarrow \vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r_0}}$

Coordinat-wise, once we write $\vec{n} = \langle a,b,c\rangle$, we obtain a

scalar eq-n of the plane through $P_0(x_0,y_0,z_0)$ with normal vector $\vec{n} = \langle a,b,c\rangle$: $\boxed{a\cdot(x-x_0)+b\cdot(y-y_0)+c(z-z_0)=0}$

Finally, setting $d := -ax_0 - by_0 - cz_0$, we obtain a

linear eq-n of the plane: $\boxed{ax+by+cz+d=0}$

Ex 4: Find an eq-n of the plane through the point $(-1,3,-4)$ and perpendicular to the line L: $x=-1-t,\ y=2+3t,\ z=-10t$.

Ex 5: Find an eq-n of the plane through the points $P(1,0,-1)$, $Q(2,1,-3)$, $R(3,-1,5)$.

(*Hint*: Find the normal vector by computing e.g. $\overrightarrow{PQ}\times\overrightarrow{PR}$)

Ex 6: Find the point at which the line L: $x=-1-t,\ y=2+3t,\ z=-10t$ intersects the plane $2x+y-\frac{z}{5}-1=0$.

(*Hint*: Plug x,y,z expressed via t into the eq-n of the plane).

Let us now consider two planes. There are two situations which can happen:

(1) the planes are parallel ($\Leftrightarrow$ their normal vectors are parallel)

(2) the planes are not parallel $\Rightarrow$ their intersection is a line L

In this case there are two standard q-s to ask:

(a) Find the angle b/w planes (which is the acute angle b/w their normal vectors)

(b) Find an equation for the line L.

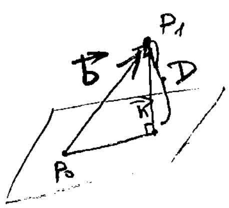
Ex 7: (a) Find an angle between the planes $x-y+z=2$ and $2x+y-2z=1$.

(b) Find symmetric and parametric eq-s for the line of their intersection.

(*Hint*: Recall that $\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}$ for part (a)

For part (b), suffices to find any point on this line and compute $\vec{n}_1\times\vec{n}_2$)

Finally, one can also compute a distance from a point to a plane. Let $P(x_1,y_1,z_1)$ be any point in $\mathbb{R}^3$, plane: $ax+by+cz+d=0$.



Choose any point P_0 in the plane.

$$D=|\mathrm{comp}_{\vec{n}}\vec{b}|=\frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}=\frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}.$$

<u>So</u>: $$D=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

Lecture #3 09/07/2017.

As we ran out of time at the end of today's lecture, let me provide a solution for Ex7.

Solution to Exercise 7

(a) First, we determine the normal vectors to the given two planes just by reading off the coefficients of x, y, z in the corresponding linear equations:
$\vec{n_1} = \langle 1, -1, 1 \rangle$, $\vec{n_2} = \langle 2, 1, -2 \rangle$.

Then, if θ is an angle between $\vec{n_1}$ and $\vec{n_2}$, we have

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|} = \frac{1\cdot 2 + (-1)\cdot 1 + 1\cdot(-2)}{\sqrt{3}\cdot\sqrt{9}} = \frac{-1}{3\sqrt{3}}$$

However: $\frac{\pi}{2} < \arccos\left(-\frac{1}{3\sqrt{3}}\right) < \pi$ and hence, the angle φ between the given two planes is the complementary:

$$\varphi = \pi - \theta \Rightarrow \boxed{\varphi = \arccos\left(\frac{1}{3\sqrt{3}}\right)}$$

! In general, if $\vec{n_1}, \vec{n_2}$ are normal vectors to the given two planes, then the angle between these planes is $\boxed{\varphi = \arccos\left(\left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}\right|\right)}$

(b) To apply the f-las from the beginning of today's class we need to

(1) find a point on the line of intersection of our two planes

(2) find a direction vector of this line.

To solve (1), we just need to come up with 3 numbers: x, y, z, which satisfy $x - y + z = 2$ and $2x + y - 2z = 1$. There are many ways to do this, e.g. set $z = 0$, so that everything boils down to $\begin{cases} x - y = 2 \\ 2x + y = 1 \end{cases}$. Adding these 2 eq-s we get $3x = 3 \Rightarrow x = 1$.

$x - y = 2 \Rightarrow y = x - 2 \overset{x=1}{\Longrightarrow} y = -1$. Hence we can pick a pt $P_0(1, -1, 0)$ on our line.

To solve (2), it suffices to note that the vector $\vec{n_1}\times\vec{n_2}$ is perpendicular to both $\vec{n_1}$ and $\vec{n_2}$, and therefore must be parallel to our line of intersection. In other words, we can choose $\vec{v} = \vec{n_1}\times\vec{n_2}$

Lecture #3 09/07/2017

(Continuation of Solution)

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \vec{i}\cdot 1 - \vec{j}\cdot(-4) + \vec{k}\cdot 3 = \langle 1, 4, 3 \rangle.$$

Hence, we can write down a parametric eq-n of our line as

$$\boxed{x = 1 + t, \; y = -1 + 4t, \; z = 3t}$$

while the symmetric eq-n takes the form

$$\boxed{\frac{x-1}{1} = \frac{y+1}{4} = \frac{z}{3}}$$

This completes the solution of Ex 7 ■

Ex 8: Compute a distance from the point $P(1, 2, -3)$ to the plane $x - 2y + 3z = 8$.

Solution of Ex 8

► As we know the distance D is given by the f-la:

$$D = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$ (see page 3)

Here a, b, c are coordinates of the normal vector to the plane, that is $a = 1$, $b = -2$, $c = 3$; while x_1, y_1, z_1 - coordinates of P, i.e. $x_1 = 1$, $y_1 = 2$, $z_1 = -3$. Finally, rewriting eq-n of the plane as $x - 2y + 3z - 8 = 0$, we find $d = -8$.

So: $$D = \frac{|1\cdot 1 + (-2)\cdot 2 + 3\cdot(-3) + (-8)|}{\sqrt{1^2 + (-2)^2 + 3^2}} = \boxed{\frac{20}{\sqrt{14}}}$$

■