Lecture #4

09/12/2017

· **Remind**:
- Office hours: Tue, Wed $2^{00} - 3^{30}$ pm
- Peer tutors: additional help
- Lecture notes will be posted on a regular basis

Last time:

Studied equations of lines and planes, and discussed how to compute their various eq-s

* **Line**:
(1) $\vec{r} = \vec{r_0} + t\cdot\vec{v}$ — "vector eq-n"
(2) $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$ — "parametric eq-n" $(\vec{v} = \langle a,b,c\rangle)$
(3) $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ — "symmetric eq-n".

* **Plane**:
(1) $\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r_0}$ — "vector eq-n"
(2) $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ — "scalar eq-n" $(\vec{n} = \langle a,b,c\rangle)$
(3) $ax + by + cz + d = 0$ — "linear eq-n" $(d = -ax_0 - by_0 - cz_0)$

* Given two points A, B in $\mathbb{R}^3$ (or $\mathbb{R}^2$), we determine the eq-n of a line passing through them by choosing $\vec{v} = \vec{AB}$ or $\vec{BA}$, while P_0 can be chosen A or B.

* Given three points P, Q, R in $\mathbb{R}^3$ not lying on the same line, we determine the eq-n of the plane containing them by choosing P_0 to be either P, Q, R, while $\vec{n}$ can be chosen as $\vec{PQ}\times\vec{PR}$ (or any other crossproduct of vectors b/w these 3 points).

* Two lines in $\mathbb{R}^3$ can be either parallel, intersect at 1 pt, or skew.
(recall how to determine which case applies)

* Line & Plane: there are 3 cases
- line belongs to the plane
- line is parallel to the plane, but does not belong to it
- line intersects the plane at 1 pt

(recall how to determine which)

* Two planes
- parallel
- intersect at a line

(recall how to compute the eq-n of a line of intersection).

* Distance from $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by $\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$

Lecture #4

03/12/2017

- Today: Vector functions (Chapter 13)

A vector function is a function whose domain is (a subset of) $\mathbb{R}$, while its range is a set of vectors.

In this class, we are most interested in vector f-s whose values are el-s of $\mathbb{R}^3$.

In other words:

$$\boxed{t \longmapsto \vec{r}(t) = \langle \underbrace{f(t), g(t), h(t)}_{\text{component functions}} \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \in \mathbb{R}^3}$$

Rmk: The domain of $\vec{r}$ consists of all $t \in \mathbb{R}$, s.t. components of $\vec{r}(t)$ are defined.

- Limits and Continuity

The limit of a vector function $\vec{r}$ is defined component-wise:

if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

if all 3 limits exist, and does not exist otherwise.

Def: A vector function $\vec{r}$ is continuous at $a \in \mathbb{R}$ if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$.

Note: $\vec{r}$ is continuous at a iff all 3 components of $\vec{r}$ are continuous at a.

Ex 1: Find $\lim_{t \to 0} \vec{r}(t)$, where $\vec{r}(t) = \langle \frac{1}{t+1}, \frac{e^t - 1}{t}, \frac{\sin t}{10t} \rangle$

► $\lim_{t \to 0} \frac{1}{t+1} = \frac{1}{0+1} = 1$

$\lim_{t \to 0} \frac{e^t - 1}{t} \underset{\text{L'Hospital Rule}}{=\!=\!=} \lim_{t \to 0} \frac{(e^t-1)'}{t'} = \lim_{t \to 0} \frac{e^t}{1} = e^0 = 1$ (could also use $e^t = 1 + t + \frac{t^2}{2!} + \dots$)

$\lim_{t \to 0} \frac{\sin t}{10t} \underset{\text{L'Hospital}}{=\!=\!=} \lim_{t \to 0} \frac{\cos t}{10} = \frac{1}{10}\cos(0) = \frac{1}{10}$

So: $\boxed{\lim_{t \to 0} \vec{r}(t) = \langle 1, 1, \frac{1}{10} \rangle}$ ■

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- Space Curves

Let f, g, h be continuous real-valued functions on an interval I. Then the locus C of points $(x,y,z) \in \mathbb{R}^3$, $x=f(t)$, $y=g(t)$, $z=h(t)$, $t \in I$, is called a space curve

Here t is called a parameter, while the above eq-s are parametric eq-s of a curve. We can think of this as a particle travelling in $\mathbb{R}^3$, where t is a time, while C stays for its trajectory.

Note: Any continuous vector f-n $\vec{r}$ defines a space curve C, traced by the tip of $\vec{r}(t)$.

Ex2: Describe the curve defined by the vector f-s:

(a) $\vec{r}(t) = \langle 1+t, 3-10t, 5+7t \rangle$

(b) $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$

(c) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

(a) Line passing through $(1,3,5)$ in the direction of $\langle 1,-10,7 \rangle$

(b) Unit circle in the xy-plane

(c) Helix, whose projection on xy-plane is the unit circle.

! Attention: You are expected to draw approximate pictures.

Ex3: Find a vector f-n that represents:

(a) a line segment b/w $P(1,0,1)$ and $Q(-2,10,3)$

(b) the curve of intersection of the cylinder $x^2+y^2=4$ and the plane $x+2y+z=3$

(a) $\vec{r}(t) = (1-t)\cdot\langle 1,0,1 \rangle + t\cdot\langle -2,10,3 \rangle = \langle 1-3t, 10t, 1+2t \rangle$

(b) the projection of this curve onto xy-plane is a circle of radius 2 centered at the origin. Hence, we can write

$x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$

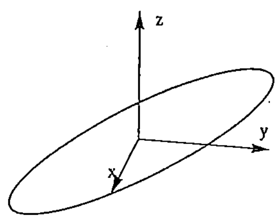
Find z component from $z = 3-x-2y = 3-2\cos t-4\sin t$.

So: $\vec{r}(t) = 2\cos t\cdot\vec{i} + 2\sin t\cdot\vec{j} + (3-2\cos t-4\sin t)\cdot\vec{k}$

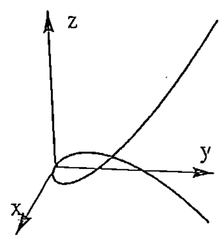
Ex4: HAND OUT MATCHING worksheet.

Matching game

1. ___ $\vec{r}(t) = (\cos(t), \sin(t), \sin(4t))$
2. ___ $\vec{r}(t) = (\cos(2t), \sin(2t), \sin(6t))$
3. ___ $\vec{r}(t) = \left(\sin(t), \cos(t), \frac{1}{2}\cos(t)\right)$
4. ___ $\vec{r}(t) = \left(t, t^2, t^3\right)$
5. ___ $\vec{r}(t) = \left(t, t^2, e^t\right)$
6. ___ $\vec{r}(t) = \left(t, t^2, \frac{1}{1+t^2}\right)$



A



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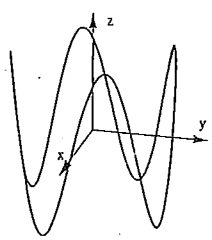

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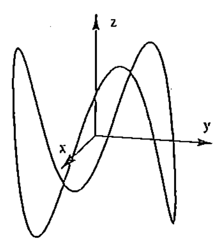

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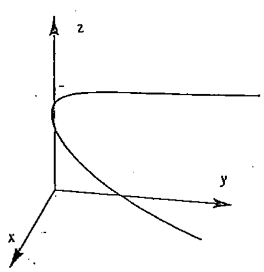
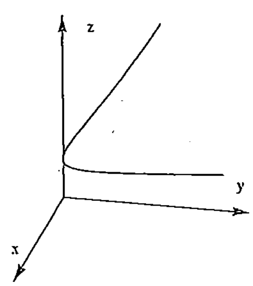


E



F

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· Derivatives (of vector functions)

Formally, the definition of the derivative $\vec{r}\,'$ of a vector function $\vec{r}$ is given in the same way as for real-valued f-s:

$$\boxed{\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}}$$

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then we immediately see that $\vec{r}\,'$ is computed component-wise:

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}}$$

Geometrically: (1) The vector $\vec{r}\,'(t)$ is tangent to the space curve (determined by $\vec{r}$ at the point $\vec{r}(t)$)

(2) If we think of $\vec{r}(t)$ as a coordinate of the particle moving along the trajectory C, then the magnitude of $r'(t)$ is nothing else than a speed of the particle.

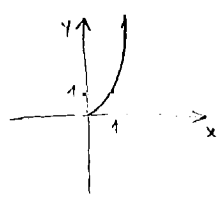
Ex 5: (a) Sketch the plane curve with the following parametric eq-n: $\begin{cases} x = e^t \\ y = e^{3t} \end{cases}$

(b) Find $r'(t)$.

(c) Find the unit tangent vector at the point where $t=0$.

(d) Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}\,'(t)$ for $t=0$.

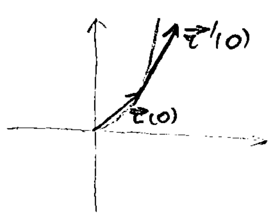
(a) $y = x^3$, $x > 0$



(b) $r'(t) = \langle e^t, 3e^{3t} \rangle$

(c) $r'(0) = \langle e^0, 3e^0 \rangle = \langle 1, 3 \rangle \Rightarrow T(0) = \frac{r'(0)}{|r'(0)|} = \frac{\langle 1,3 \rangle}{\sqrt{10}} = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}$

(d)



Ex6: Find the parametric eq-n for the tangent line to the curve $x = \ln(t^2+1)$, $y = \sqrt{t^2+4}$, $z = e^{2t}\cdot \cos t$ at the point $(0,2,1)$.

► Point $(0,2,1)$ corresponds to $t=0$.

$\vec{r}\,'(t) = \langle \ln(t^2+1)', \sqrt{t^2+4}\,', (e^{2t}\cos t)' \rangle$

Here: $\ln(t^2+1)' = \frac{1}{t^2+1}\cdot 2t$

$\sqrt{t^2+4}\,' = \frac{1}{2\sqrt{t^2+4}}\cdot 2t = \frac{t}{\sqrt{t^2+4}}$

$(e^{2t}\cos t)' = 2e^{2t}\cos t - e^{2t}\sin t$

$\Rightarrow \vec{r}\,'(t) = \langle \frac{2t}{t^2+1}, \frac{t}{\sqrt{t^2+4}}, 2e^{2t}\cos t - e^{2t}\sin t \rangle$

So $\vec{r}\,'(0) = \langle 0,0,2 \rangle$.

Therefore, the tangent line is $x=0$, $y=2$, $z=1+2t$. ■

Properties

Please look at the table (Theorem 3) on p. 858 of your textbook. The most interesting are:

$$\boxed{\begin{aligned} [\vec{u}(t)\cdot\vec{v}(t)]' &= \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t) \\ [\vec{u}(t)\times\vec{v}(t)]' &= \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t) \end{aligned}}$$

· **Integrals** (of a vector function)

Formally, the integral of a vector-valued function is defined in the same way as for real-valued f-s. On the technical/computational side, if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then:

$$\boxed{\int_a^b \vec{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \rangle \in \mathbb{R}^3}$$

Ex 7: Evaluate the integrals

(a) $\int_2^5 (t^2\vec{i} + \frac{1}{t+2}\vec{j} + e^{3t}\vec{k})\,dt$

(b) $\int_0^1 (te^{t^2}\vec{i} + \sin t\cos t\,\vec{j})\,dt$

► (a) $\int_2^5 t^2\,dt = \frac{t^3}{3}\Big|_2^5 = \frac{125-8}{3} = \frac{117}{3}$, $\int_2^5 \frac{1}{t+2}\,dt = \ln|t+2|\Big|_2^5 = \ln\left(\frac{7}{4}\right)$, $\int_2^5 e^{3t}\,dt = \frac{e^{3t}}{3}\Big|_2^5 = \frac{e^{15}-e^6}{3}$

So: Get $\boxed{\frac{117}{3}\vec{i} + \ln\left(\frac{7}{4}\right)\vec{j} + \frac{e^{15}-e^6}{3}\vec{k}}$

(b) $\int_0^1 te^{t^2}\,dt = \frac{e^{t^2}}{2}\Big|_0^1 = \frac{e-1}{2}$, $\int_0^1 \sin t\cos t\,dt = \frac{\sin^2 t}{2}\Big|_0^1 = \frac{\sin^2(1)}{2}$. So: $\boxed{\frac{e-1}{2}\vec{i} + \frac{\sin^2(1)}{2}\vec{j}}$

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- **Length of a curve**

For the case of plane curves given by $x=f(t)$, $y=g(t)$, many of you know that the length of the corresponding curve for $a\le t\le b$ equals:

$$\boxed{L=\int_a^b \sqrt{f'(t)^2+g'(t)^2}\,dt}$$

If you don't know this f-la, don't worry, as we will now need its generalization for the case of space curves.

Suppose that the space curve C is given via $\boxed{\vec{r}(t)=\langle f(t),g(t),h(t)\rangle \quad a\le t\le b}$

Then we think of the length of C as a limit of lengths of inscribed polygons. Each polygon corresponds to a collection of points $a=t_0<t_1<\dots<t_N=b$ and its length is $|\vec{r}(t_1)-\vec{r}(t_0)|+|\vec{r}(t_2)-\vec{r}(t_1)|+\dots+|\vec{r}(t_N)-\vec{r}(t_{N-1})|$.

Note that $|\vec{r}(t_{i+1})-\vec{r}(t_i)|=\sqrt{(f(t_{i+1})-f(t_i))^2+(g(t_{i+1})-g(t_i))^2+(h(t_{i+1})-h(t_i))^2}$!

At the end, it is easy to see that we get the following f-la.

length of C

$$\boxed{L=\int_a^b \sqrt{(f'(t))^2+(g'(t))^2+(h'(t))^2}\,dt=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt}$$

$$\Downarrow$$

$$\boxed{L=\int_a^b |\vec{r}'(t)|\,dt}$$

Ex 8: Find the length of the following curves:

(a) $\vec{r}(t)=2\cos t\,\vec{i}+2\sin t\,\vec{j}+5t\,\vec{k}$ from the point $(2,0,-10\pi)$ to $(-2,0,5\pi)$

(b) $\vec{r}(t)=6t\cdot\vec{i}+t^3\cdot\vec{j}+3t^2\cdot\vec{k}$, $-1\le t\le 1$.

(a) $L=\int_{-2\pi}^{\pi}\sqrt{(-2\sin t)^2+(2\cos t)^2+5^2}\,dt=\int_{-2\pi}^{\pi}\sqrt{29}\,dt=3\sqrt{29}\cdot\pi$

(b) $L=\int_{-1}^{1}\sqrt{6^2+(3t^2)^2+(6t)^2}\,dt=\int_{-1}^{1}\sqrt{36+36t^2+9t^4}\,dt=$

$=\int_{-1}^{1}3\sqrt{t^4+4t^2+4}\,dt=\int_{-1}^{1}3(t^2+2)\,dt=(t^3+6t)\Big|_{-1}^{1}=7-(-7)=14$

! In part (a), the limits of integration are found by solving $5t=-10\pi$ and $5t=5\pi$, respectively.