Lecture #5

09/14/2017

Last time

Last time we discussed vector functions and saw the natural relation between vector functions and space curves.

On the technical side, we saw that one can define

$$\boxed{\lim_{t\to a} \vec{r}(t),\ \vec{r}'(t),\ \int_a^b \vec{r}(t)dt}$$

in the same way as for real-valued functions.

Moreover, each of them is computed component-wise.

In other words, given $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, we get

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t)\rangle$$

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

$$\int_a^b \vec{r}(t)dt = \langle \int_a^b f(t)dt,\ \int_a^b g(t)dt,\ \int_a^b h(t)\,dt \rangle$$

! When asked to match space curves with vector functions, our strategy is first to check if certain points in $\mathbb{R}^3$ belong to the curve and to the values of vector f-s, e.g. origin, pts on axes. Second, we study the projections on coordinate planes.

Rmk 1: $\vec{r}(t)$ is called continuous at $a\in\mathbb{R}$ if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.
Clearly, $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at a if f, g, h are continuous at a.

Rmk 2: The vector $\vec{r}'(t)$ is tangent to the space curve C determined by $\vec{r}$ at the point $\vec{r}(t)$. Moreover, the magnitude $|\vec{r}'(t)|$ is just the speed of a particle travelling along C according to $\vec{r}$.

Ex1 (=Ex6 from Lecture 4): Find the parametric eq-n for the tangent line to the curve $x = \ln(t^2+1)$, $y = \sqrt{t^2+4}$, $z = e^{2t}\cdot\cos t$ at the point $(0,2,1)$.

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The length of a planar curve $x=f(t)$, $y=g(t)$, $a\le t\le b$ is given by

$$\boxed{L=\int_a^b \sqrt{f'(t)^2+g'(t)^2}\,dt} \quad (1)$$

The length of a space curve $x=f(t)$, $y=g(t)$, $z=h(t)$, $a\le t\le b$, equals

$$\boxed{L=\int_a^b \sqrt{f'(t)^2+g'(t)^2+h'(t)^2}\,dt} \quad (2)$$

Rmk1: Both formulas can be uniformly written as

$$\boxed{L=\int_a^b |\vec{r}\,'(t)|\,dt} \quad (3)$$

! Note that this f-la has an obvious geometric meaning:
"the total distance a particle travels equals the integral of its speed".

Rmk2: In the case $x=t$, f-la (1) reads as $\boxed{L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$ (4), which should look familiar to most of you.

! Actually, one can derive (1) from (4) as follows:
make change of variable $x=f(t) \Rightarrow dx=\frac{dx}{dt}dt=f'(t)dt$
Then $\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot\frac{dx}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$, due to chain rule.

Ask if there are any q-s in regards to Tuesday Lecture.

- Velocity & Acceleration (see Sect. 13.4)

Imagine a particle moving in $\mathbb{R}^3$, whose position at time t is given by $\vec{r}(t)$. The velocity vector $\vec{v}(t)$ can be thought of as a limit of the ratio of a displacement vector per unit of time, i.e.

$$\boxed{\vec{v}(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h} = \vec{r}\,'(t)}$$

Def: The speed of a particle at time t is the magnitude $|\vec{v}(t)| = |\vec{r}\,'(t)|$.

[! This is intuitively clear as $|\vec{v}(t)| = |\vec{r}\,'(t)|$ = rate of change of distance, due to the f-la $L = \int_a^b |\vec{r}\,'(t)|dt$]

Def: The acceleration of the particle is defined as the derivative of velocity:

$$\boxed{\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)}$$

Ex 2: Find the velocity, acceleration, and speed of a particle with the position vector $\vec{r}(t) = \langle t^{11}, e^{t^2}, \ln(t^2+1)\rangle$

- $\vec{v}(t) = \langle 11t^{10}, 2t\cdot e^{t^2}, \frac{2t}{t^2+1}\rangle$

$\vec{a}(t) = \langle 110t^9, (4t^2+2)e^{t^2}, \frac{2(t^2+1)-2t\cdot 2t}{(t^2+1)^2}\rangle$

$|\vec{v}(t)| = \sqrt{121t^{20} + 4t^2e^{2t^2} + \frac{4t^2}{(t^2+1)^2}}$ ∎

Ex 3: A moving particle starts at an initial position $\vec{r}(0) = \langle -2,3,5\rangle$. Its velocity is $\vec{v}(t) = \sin t\cdot\vec{i} - te^{t^2}\cdot\vec{j} + \sin^3 t\cos t\cdot\vec{k}$.
Find the position at time t.

- $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)du = \langle -2,3,5\rangle + \int_0^t \langle \sin t, -te^{t^2}, \sin^3 t\cos t\rangle dt$

$= \langle -2,3,5\rangle + \langle 1-\cos t, \frac{1-e^{t^2}}{2}, \frac{\sin^4 t}{4}\rangle$

$= \langle -1-\cos t, 3\frac{1}{2} - \frac{1}{2}e^{t^2}, 5 + \frac{\sin^4 t}{4}\rangle$ ∎

! Analogously, given $\vec{r}(0)$, $\vec{v}(0)$, $\vec{a}(t)$, we can recover $\vec{v}(t)$ and $\vec{r}(t)$ via

$$\boxed{\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)du, \quad \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)du}$$

Ex 4: A moving particle starts at an initial position $\vec{r}(0) = <-1, 0, 2>$ with initial velocity $\vec{v}(0) = <1, 2, -3>$ and acceleration $\vec{a}(t)$ given by $\vec{a}(t) = e^t \cdot \vec{i} - 2t \cdot \vec{j} + \sin t \cdot \vec{k}$.

Find its velocity and position at time t.

$\vec{a}(t) = \vec{v}'(t) \Rightarrow \vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du$.

Know: $\int_0^t e^u du = e^t - 1$; $\int_0^t -2u\,du = -u^2\Big|_0^t = -t^2$; $\int_0^t \sin u\,du = -\cos u\Big|_0^t = 1 - \cos t$.

So: $\vec{v}(t) = <1, 2, -3> + <e^t - 1, -t^2, 1 - \cos t> = \boxed{<e^t, 2 - t^2, -2 - \cos t>}$

Likewise, $\vec{v}(t) = \vec{r}'(t) \Rightarrow \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du$

Know: $\int_0^t e^u du = e^u\Big|_0^t = e^t - 1$; $\int_0^t (2 - u^2)\,du = \left(2u - \frac{u^3}{3}\right)\Big|_0^t = 2t - \frac{t^3}{3}$;

$\int_0^t (-2 - \cos u)\,du = (-2u - \sin u)\Big|_0^t = -2t - \sin t$.

So: $\vec{r}(t) = <-1, 0, 2> + <e^t - 1, 2t - \frac{t^3}{3}, -2t - \sin t>$

$= \boxed{<e^t - 2, 2t - \frac{t^3}{3}, 2 - 2t - \sin t>}$

Answer: $\vec{v}(t) = <e^t, 2 - t^2, -2 - \cos t>$, $\vec{r}(t) = <e^t - 2, 2t - \frac{t^3}{3}, 2 - 2t - \sin t>$

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- Our next & main topic: "Functions of several variables"

We will start by considering the first nontrivial case, i.e. functions of two variables.

Def: A function of two variables is a function whose domain D is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$

Convention: If a function f is given by the formula and no domain is specified, then the domain of f is the set of all pairs $(x,y) \in \mathbb{R}^2$, s.t. $f(x,y)$ is a well-defined real number, while the range of f is the set of all possible values $f(x,y)$.

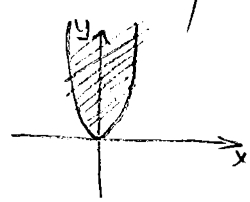
Ex 5: Sketch the domain for each of the following f-s:

(a) $f(x,y) = \sqrt[6]{y - x^6}$

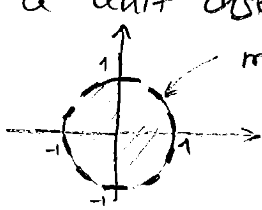
(b) $f(x,y) = \ln(1 - x^2 - y^2)$

(c) $f(x,y) = \dfrac{\sqrt{\sin x}}{y}$

(a) $y - x^6 \geq 0 \Leftrightarrow y \geq x^6$. The corresponding domain looks as:



(b) $1 - x^2 - y^2 > 0 \Leftrightarrow x^2 + y^2 < 1 \Leftrightarrow$ distance from $(0,0)$ to (x,y) is less than 1. Hence, we get a unit disk without boundary



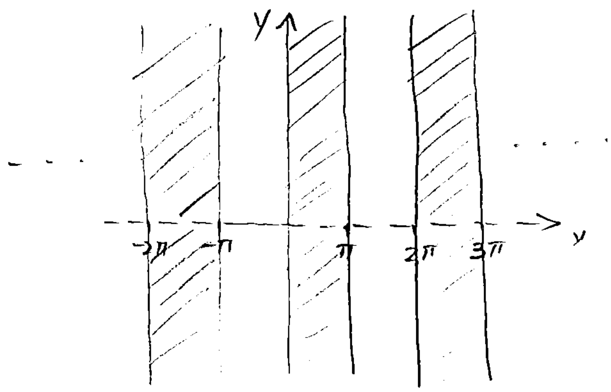
make the boundary "dotted" to show it is not included

(c) There are two conditions in this case: $y \neq 0$ and $\sin x \geq 0$.

Recalling graph of $\sin x$, $\sin x \geq 0$ iff $x \in \ldots \cup [-2\pi, -\pi] \cup [0, \pi] \cup [2\pi, 3\pi] \cup \ldots$

- (Continuation of Ex5(c))

Hence, the corresponding domain is a union of vertical strips with points on x-axis excluded



Ex6: Find the range of each of 3 functions of Ex5.

- (a) $[0, +\infty)$
  (b) $(-\infty; 0]$
  (c) $\mathbb{R} = (-\infty, +\infty)$

Pretty clear, but you are expected to provide an argument

- Graphs and Level Curves

The most common way to visualize a function of two variables is by considering the corresponding graph.

Def: If f is a function of two variables with domain D, then the graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$, s.t. $(x, y) \in D$ and $z = f(x, y)$

[Compare to the notion of a graph of function of one variable].

An alternative way to visualize $f(x,y)$ is by drawing level curves.

Def: The level curves of a function f of two variables are the curves with equation $f(x,y) = k$, where k is a constant (in the range of f)

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Ex 7: Sketch the graphs of the following functions:

(a) $f(x,y) = 2 - x - y$

(b) $f(x,y) = \sqrt{4 - x^2 - y^2}$

(c) $f(x,y) = x^2 + 9y^2$

(d) $f(x,y) = \sqrt{1 - y^2}$

(e) $f(x,y) = 2\sqrt{x^2 + y^2}$.

Draw the level curves in each of the above 5 cases.

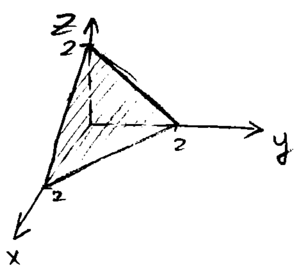
(a) Want to draw a surface given by $z = 2 - x - y$

$z = 2 - x - y \iff x + y + z = 2$.

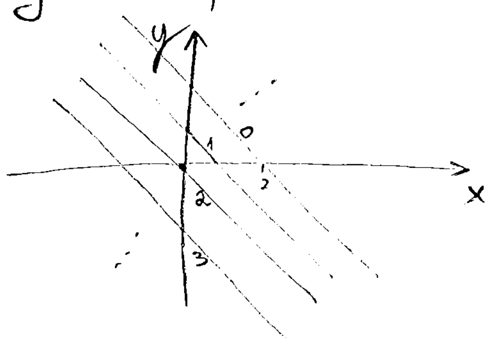
We know that the latter is an equation of a plane. To graph this plane, we find the intercepts (i.e. intersections with x, y, z-axes). To find x-intercept, set $y = z = 0 \Rightarrow x + 0 + 0 = 2 \Rightarrow x = 2$.

Analogously, y-intercept and z-intercept are also 2.

Thus, we can sketch part of the plane, which is a triangle whose vertices are exactly intercepts:



The corresponding level curves are given by $k = 2 - x - y \iff x + y = 2 - k$. Hence they are parallel lines depicted below:



! The numbers over level curves keep track of corresponding number k.

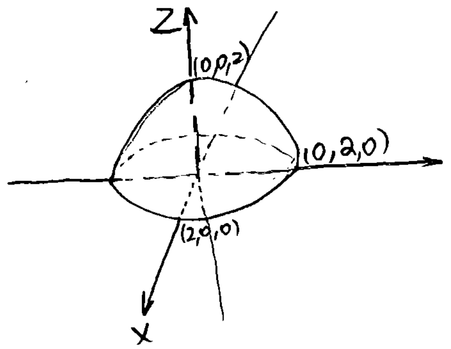
- (Continuation of Ex 7)

(b) $z=\sqrt{4-x^2-y^2} \Rightarrow z^2=4-x^2-y^2 \Leftrightarrow \underbrace{x^2+y^2+z^2=4.}$

this equation determines a sphere of radius 2 with a center at the origin.

However: first arrow is not an equivalence, i.e. $z^2=4-x^2-y^2 \not\Rightarrow z=\sqrt{4-x^2-y^2}$.
as $-\sqrt{4-x^2-y^2}$ also satisfies $z^2=4-x^2-y^2$.

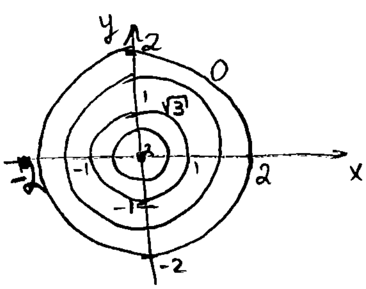
Hence, we need to take only part of sphere with $z\geq 0$.



or something in this spirit...

The level curves are given by $k=\sqrt{4-x^2-y^2} \Leftrightarrow \begin{cases} k\geq 0 \\ x^2+y^2=4-k^2 \end{cases}$

Hence, we get a collection of circles centered at the origin of radius ≤ 2 (not that we get a pt $=(0,0)$ viewed as a circle of radius 0)

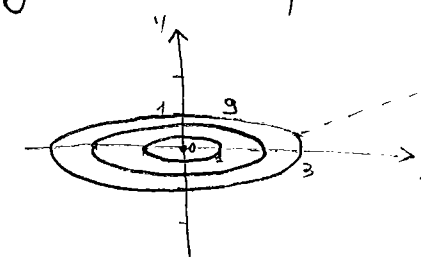


(c) $z=x^2+9y^2$.

Let us start by drawing level lines. The range is obviously $[0,+\infty)$

For $k=0$, the corresponding level line is just a point $(0,0)$

For $k>0$, we get an ellipse. Hence the level curves look as:

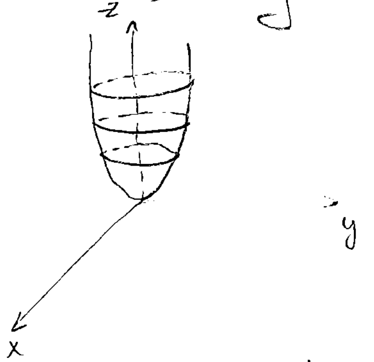


collection of scaled version of the same ellipse of an arbitrary big scale.

► (Continuation of Ex 7)

(c) On the other hand, intersections with yz and xz planes are parabolas $z = 9y^2$ and $z = x^2$, respectively.

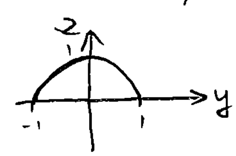
Combining all this information, we can draw a final picture



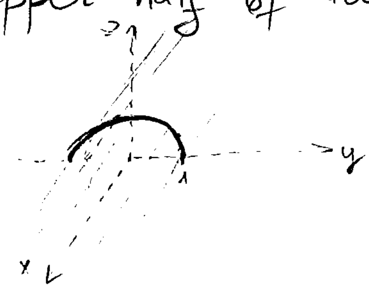
This is called "elliptic paraboloid"

(d) $z = \sqrt{1-y^2} \Longleftrightarrow \begin{cases} z \geq 0 \\ z^2 = 1 - y^2 \end{cases} \Longleftrightarrow \begin{cases} z \geq 0 \\ y^2 + z^2 = 1 \end{cases}$

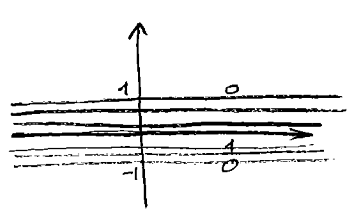
The latter does not depend on x at all, hence, we will get the cone over the intersection of this graph with yz-plane. The latter is the upper half of the unit circle



Hence, we get



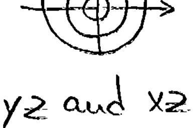
The level curves are defined via $k = \sqrt{1-y^2} \Leftrightarrow \begin{matrix} k \geq 0 \\ y = \sqrt{1-k^2} \end{matrix}$. Hence, get



← a collection of lines parallel to the x-axis at the height $\in [0,1]$.

e) $z = 2\sqrt{x^2+y^2}$

The level curves are given by $k = 2\sqrt{x^2+y^2} \Leftrightarrow \begin{cases} k \geq 0 \\ x^2+y^2 = k^2/4 \end{cases}$ — circles centered at the origin



Intersections with yz and xz planes are given by $z = 2|y|$ and $z = 2|x|$, resp.

So: the graph will be a cone