Lecture #7

- **Last time**

Last time we discussed how to compute $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ or prove that this limit does not exist.

Two key points:

(1) If you can come up with two curves passing through (x_0,y_0) such that the limits of f along these curves at point (x_0,y_0) are different, you may conclude that the limit does not exist.

We usually try lines, parabolas, etc.

However: if all the limits are the same you CAN NOT conclude that the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

(2) Our main tool to prove that the limit exists is the squeeze Thm.

Most tricky part: Find appropriate functions $g(x,y)$, $h(x,y)$

! If you have expressions involving x^2+y^2 it may be a good idea to switch to polar coordinates.

Ex1: Do the following limits exist (and if yes, then compute them):

(a) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ (switch to polar coordinates $= \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1$.)

(b) $\lim_{(x,y)\to(2,1)} \frac{(y-1)^5}{e^{x-2}+(y-1)^4}$ (Note that the function is well-defined at $x=2, y=1$: $\frac{0}{e^0+0} = \frac{0}{1} = 0$. As we will explain on page 2 this is a leginete answer)

(c) $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$ (Along the line $x=y=0$, the limit is zero while along the line $x=y, z=0$, the limit is $\frac{1}{2}$)

Note: In part (c) we have a function in 3 variables, but we follow the same strategy.

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• Continuous functions

Def: A function f of two variables is called continuous at (a,b) if
$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$
A function f is continuous on D if it is continuous at every $(a,b)\in D$

Ex2: Determine the set of points at which the function is continuous:

(a) $f(x,y) = \dfrac{e^{\sin(xy^2)}}{e^{e^y} + \ln(1+y^{10}x^6)}$

(b) $f(x,y) = \dfrac{e^x + e^y}{e^{xy}-1}$

(c) $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$

– (a) As polynomials are continuous, while exponent, sin are continuous everywhere and ln is continuous on $(0,+\infty)$, we see that numerator and denominator are continuous.

As $y^{10}x^6 \geq 0 \Rightarrow \ln(1+y^{10}x^6)\geq 0$; $e^{e^y} > 0 \Rightarrow$ denominator is always $\neq 0 \Rightarrow$ $\boxed{f \text{ is continuous at all } (x,y)\in\mathbb{R}^2.}$

(b) As in (a), numerator & denominator are continuous.

However, denominator may equal to zero:

$e^{xy}-1 = 0 \Leftrightarrow xy=0 \Leftrightarrow x=0$ or $y=0$.

However, numerator is never zero.

$\boxed{\text{So, } f \text{ is continuous at } \{(x,y)\in\mathbb{R}^2 \mid x\neq0, y\neq0\}}$

(c) Clearly (as above) f is continuous at all $(x,y)\neq(0,0)$.

On the other hand, $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3}{2x^2+y^2} = 0$ (by squeeze thm: work it out!)

As $f(0,0)=0$, we see that f is also continuous at $(0,0)$.

$\boxed{\text{So, } f \text{ is continuous at all } (x,y)\in\mathbb{R}^2}$

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• Partial derivatives

Given a function $f(x,y)$ of two variables, one can consider partial derivatives at a point (a,b), denoted $f_x(a,b)$ and $f_y(a,b)$:

$$\boxed{\begin{aligned} f_x(a,b) &= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} \\ f_y(a,b) &= \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h} \end{aligned}}$$

In other way, to compute $f_x(a,b)$, we set $y=b$ into $f(\cdot,\cdot)$ to obtain a function of 1 variable x: $g(x):=f(x,b)$, and then take the usual derivative of $g(x)$ w.r.t. x.

Similarly, to compute $f_y(a,b)$, we set $h(y):=f(a,y)$ and compute the usual derivative of $h(y)$ w.r.t. y.

Ex 3: (a) If $f(x,y)=\sin(xy)+e^{y^2x}$, find $f_x(1,0)$ and $f_y(1,0)$.

(b) If $f(x,y,z)=z^2\cos(x+e^y)+e^{2x}\cdot\sqrt{y^2+z^4}$, find $f_x(1,1,1)$, $f_y(1,1,1)$, $f_z(1,1,1)$

(a) $f_x(x,y)=\cos(xy)\cdot y+e^{y^2x}\cdot y^2 \Rightarrow f_x(1,0)=0+0=0$

$f_y(x,y)=\cos(xy)\cdot x+e^{y^2x}\cdot 2yx \Rightarrow f_y(1,0)=1$

(b) $f_x(x,y,z)=-z^2\cdot\sin(x+e^y)+e^{2x}\cdot 2\cdot\sqrt{y^2+z^4} \Rightarrow f_x(1,1,1)=-\sin(e+1)+2\sqrt{2}e$

$f_y(x,y,z)=-z^2\cdot\sin(x+e^y)\cdot e^y+e^{2x}\cdot\frac{1}{2\sqrt{y^2+z^4}}\cdot 2y \Rightarrow$

$\Rightarrow f_y(1,1,1)=-e\sin(e+1)+\frac{1}{\sqrt{2}}e^2$

$f_z(x,y,z)=2z\cos(x+e^y)+e^{2x}\cdot\frac{1}{2\sqrt{y^2+z^4}}\cdot 4z^3 \Rightarrow$

$\Rightarrow f_z(1,1,1)=2\cos(e+1)+\sqrt{2}\cdot e^2$ ∎

! The case of functions in more than 2 variables is treated completely analogously

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Note that if we intersect the graph of the function $f(x,y)$ with plane $y=b$, then inside this plane we get a graph of the function $g(x)$ from previous page and hence $f_x(a,b) = g'_x(a)$ is a slope of tangent line to this graph over $x=a$.

Similarly, if we intersect with the plane $x=a$, then we get a graph of the function $h(y)$ from the previous page and hence $f_y(a,b) = h'_y(b)$ is a slope of the tangent line to this graph over $y=b$.

Higher Derivatives

If f is a function of two variables, then f_x and f_y are also functions of two variables, so we can consider their partial derivatives as well:

$(f_x)_x,\ (f_x)_y,\ (f_y)_x,\ (f_y)_y$ – second partial derivatives of f

To simplify notations we omit ( ), so we denote them $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ resp.

Ex 4: Find the second partial derivatives of

$$f(x,y) = \sin(x^2 e^y)$$

$f_x(x,y) = \cos(x^2e^y)\cdot 2xe^y$

$f_y(x,y) = \cos(x^2e^y)\cdot x^2e^y$

$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\cos(x^2e^y)\cdot 2xe^y\right) = 2e^y\left(\cos(x^2e^y) - \sin(x^2e^y)\cdot 2xe^y\cdot x\right)$

$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\cos(x^2e^y)\cdot 2xe^y\right) = 2x\cdot\left(-\sin(x^2e^y)\cdot x^2e^{2y} + \cos(x^2e^y)e^y\right)$

$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\cos(x^2e^y)x^2e^y\right) = e^y\cdot\left(-\sin(x^2e^y)\cdot 2x\cdot x^2 + \cos(x^2e^y)\cdot 2x\right)$

$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\cos(x^2e^y)x^2e^y\right) = x^2\left(-\sin(x^2e^y)\cdot x^2e^{2y} + \cos(x^2e^y)\cdot e^y\right)$

! Note that in the above example $f_{xy} = f_{yx}$. This is not a coincidence

Thm: If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Ex 5: Verify that the function $u(x,t) = \sin(x-at)$ satisfies the wave equation $u_{tt} = a^2\cdot u_{xx}$

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- **Chain Rule**

Let me start by reminding the chain rule for functions of 1 variable.
Let $y=f(x)$, while $x=g(t)$, then we see that $y=f(g(t))$ – function of t.

Chain Rule: $\boxed{\frac{d}{dt}y = \frac{dy}{dx}\cdot\frac{dx}{dt} = f'(g(t))\cdot g'(t)}$

Now we want an analogous rule in the case when each function involved is of several variables.

Case 1: $z=f(x,y)$, while $x=g(t)$, $y=h(t)$ $\Rightarrow$ $z=f(g(t),h(t))$ – function in t.

$$\boxed{\frac{dz}{dt} = f_x(g(t),h(t))\cdot g'(t) + f_y(g(t),h(t))\cdot h'(t) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}}$$

Case 2: $z=f(x,y)$, while $x=g(s,t)$, $y=h(s,t)$ $\Rightarrow$ $z=f(g(s,t),h(s,t))$ – function on s,t

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Ex6: If $z=x^3+e^{2y}$, $x=\cos(t^2)$, $y=\sin(t^3)$, find $\frac{dz}{dt}$ at $t=0$

► $\frac{dz}{dt} = 3\cos^2(t^2)\cdot(-\sin(t^2)\cdot 2t) + e^{2\sin(t^3)}\cdot 2\cdot\cos(t^3)\cdot 3t^2$

Evaluating at $t=0$, we get $\left.\frac{dz}{dt}\right|_{t=0} = 0$

! However, instead of substituting f-s in t instead of x,y, we can notice that at $t=0$: $x=1$, $y=0$ and hence:

$\frac{dz}{dt} = 3x^2\cdot(-\sin(t^2)\cdot 2t) + 2e^{2y}(\cos(t^3)\cdot 3t^2)$ ⇝ plug $t=0$, $x=1$, $y=0$

Ex7: If $z=e^{x^2}\cos(y^3)$, $x=st$, $y=s^3-t^2$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

► $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (e^{x^2}\cdot 2x\cdot\cos(y^3))\cdot t + (e^{x^2}\cdot(-\sin(y^3)\cdot 3y^2))\cdot 3s^2$

$= e^{s^2t^2}\cdot 2st\cdot\cos((s^3-t^2)^3)\cdot t - e^{s^2t^2}\cdot\sin((s^3-t^2)^3)\cdot 3(s^3-t^2)^2\cdot 3s^2$

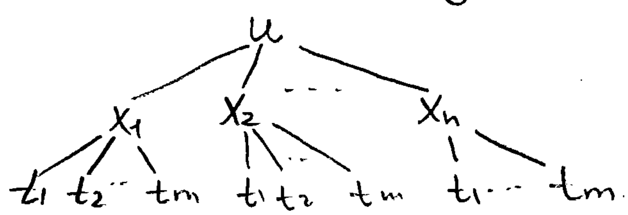
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = (e^{x^2}\cdot 2x\cdot\cos(y^3))\cdot s + (e^{x^2}\cdot(-\sin(y^3)\cdot 3y^2))\cdot(-2t)$

$= e^{s^2t^2}\cdot 2st\cdot\cos((s^3-t^2)^3)\cdot s + e^{s^2t^2}\sin((s^3-t^2)^3)\cdot 3(s^3-t^2)^2\cdot 2t$ ∎

Case 3: Let $u = u(x_1, \ldots, x_n)$, while $x_1 = f_1(t_1, \ldots, t_m), \ldots, x_n = f_n(t_1, \ldots, t_m)$. Then:

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i}}$$

We illustrate this case by drawing a tree of variables involved



Ex 8: If $u = x^2y + y^2z^3$, while $x = re^{s^2}$, $y = \sin(r+s)\cdot t^2$, $z = e^{r-s}\cdot\cos(t)$

Find the value $\frac{\partial u}{\partial s}$ when $r=1, s=0, t=1$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s} =$$

$$= (2xy)\cdot(r\cdot e^{s^2}\cdot 2s) + (x^2 + 2yz^3)\cdot(\cos(r+s)t^2) + (3y^2z^2)\cdot(e^{r-s}\cdot(-1)\cdot\cos t)$$

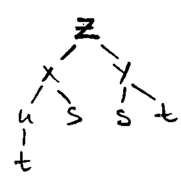
When $r=1, s=0, t=1$, we have $x = 1$, $y = \sin(1)$, $z = e\cdot\cos(1)$

So: $\frac{\partial u}{\partial s} = 2\sin(1)\cdot 0 + (1 + 2e^3\sin(1)\cos^3(1))\cdot\cos(1) - 3e^2\sin^2(1)\cos^2(1)\cdot e\cdot\cos(1)$

... simplify ...

Ex 9: If $z = e^x\sin(y)$, $x = u^2s$, $y = s^2t^3$, $u = e^t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Note that the graph is



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (e^x\sin y)\cdot u^2 + (e^x\cos y)\cdot 2st^3 =$$

$$= e^{e^{2t}\cdot s}\cdot\sin(s^2t^3)\cdot e^{2t} + e^{e^{2t}\cdot s}\cdot\cos(s^2t^3)\cdot 2st^3$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} =$$

$$= (e^x\sin y)\cdot(2us)\cdot e^t + (e^x\cos y)\cdot 3s^2t^2 =$$

$$= e^{e^{2t}\cdot s}\cdot\sin(s^2t^3)\cdot 2e^ts\cdot e^t + e^{e^{2t}\cdot s}\cdot\cos(s^2t^3)\cdot 3s^2t^2$$

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• Implicit differentiation

Suppose that y is a function in $\overset{y=f(x)}{x}$, which we do not know explicitly, but rather know that $F(x,y)=0$ for a given function F. In this case, we say that y is given implicitly.

As $F(x,y) = F(x,f(x)) = 0$ for any $x \Rightarrow \frac{d}{dx} F(x,f(x)) = 0$.

By the above chain rule

$$\frac{d}{dx} F(x,f(x)) = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

Ex 10: Find y' if $x^4 + e^y = 2x^2y^3$

This equation can be written as

$$F(x,y) := x^4 + e^y - 2x^2y^3 = 0$$

Hence: $\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{4x^3 - 4xy^3}{e^y - 6x^2y^2}$ ■

Likewise if $z = f(x,y)$ and f is not given explicitly, but rather $F(x,y,z)=0$.
Applying chain rule to differentiate $F(x,y,f(x,y))=0$, we get:

Diff. wx: $0 = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x}$

Diff. wy: $0 = \frac{\partial F}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial y}}_{0} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y}$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

Ex 11: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2 + y^4 + z^3 = 1 - 5\sin(x)e^y z$

This eq-n can be written as $F(x,y,z) := x^2 + y^4 + z^3 + 5\sin(x)e^y z - 1 = 0$

$$\frac{\partial z}{\partial x} = -\frac{2x + 5e^y z \cdot \cos(x)}{3z^2 + 5\sin(x)e^y}$$

$$\frac{\partial z}{\partial y} = -\frac{4y^3 + 5z\sin(x)e^y}{3z^2 + 5\sin(x)e^y}$$ ■