- Two organizational comments:
  - Please ignore any kind of information on Canvas regarding the homework which has not been graded yet
  - The uploaded worksheet with practice problems on limits is solely for your own practice. In particular, problems marked by * will not have their analogues on the midterm 1.

- **Chain Rule** (Section 14.5)

This part follows pages 5-7 from Lecture #7, which we did not have time to discuss in the previous class.

To save time in class, change Ex 7 to: $z = e^x\cos(y)$, $\begin{matrix} x = st \\ y = s^3 - t^2 \end{matrix}$

$$\left(\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = e^x\cos y\cdot t - e^x\sin(y)\cdot 3s^2 = e^{st}\cos(s^3-t^2)\cdot t - e^{st}\cdot\sin(s^3-t^2)\cdot 3s^2 \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = e^x\cos y\cdot s + e^x\sin(y)\cdot 2t = e^{st}\cdot\cos(s^3-t^2)\cdot s + 2te^{st}\sin(s^3-t^2) \end{aligned}\right)$$

Directional Derivatives (Section 14.6)

Let us remind the notion of partial derivatives from last class:

$$f_x(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

which basically tell how fast f changes when moving in the x- or y- directio[n]. However, it is natural to ask how fast $f(\cdot,\cdot)$ changes moving in the arbitrary direction. This admits the following simple answer.

Fix a <u>unit</u> vector $\vec{u} = \langle a, b\rangle$ and consider the restriction of function f onto the line in the direction of $\vec{u}$ passing through (x_0, y_0), i.e. consider $g(t) := f(x_0 + at, y_0 + bt)$.

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Then $g'(t)\big|_{t=0}$ is exactly the "speed of change" of f at point (x_0,y_0) in the direction of $\vec{u}$.

Def: The directional derivative of f at (x_0,y_0) in the direction of a unit vector $\vec{u}=\langle a,b\rangle$ is

$$\boxed{D_{\vec{u}}f(x_0,y_0):=\lim_{h\to 0}\frac{f(x_0+ha,\,y_0+hb)-f(x_0,y_0)}{h}} \quad (1)$$

if the limit exists

Using chain rule we see that

$$g'(t)=\frac{\partial f}{\partial x}\cdot a+\frac{\partial f}{\partial y}\cdot b \Rightarrow \boxed{D_{\vec{u}}f(x,y)=f_x(x,y)\cdot a+f_y(x,y)\cdot b} \quad (2)$$

Rmk: If $\vec{u}=\vec{i}$, then $D_{\vec{u}}f=\partial_x f$
$\vec{u}=\vec{j}$, then $D_{\vec{u}}f=\partial_y f$

Ex1: Find the directional derivative of $f(x,y)=e^x\cos y$ at the point $(0,\pi)$ in the direction indicated by the angle $\theta=-\frac{\pi}{4}$

$\theta=-\frac{\pi}{4} \Rightarrow \vec{u}=\langle a,b\rangle=\langle\cos(-\frac{\pi}{4}),\sin(-\frac{\pi}{4})\rangle=\langle\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\rangle$

$$\left.\begin{aligned}&D_{\vec{u}}f(0,\pi)=\partial_x f(0,\pi)\cdot\frac{1}{\sqrt{2}}-\partial_y f(0,\pi)\cdot\frac{1}{\sqrt{2}}\\&\partial_x f(x,y)=e^x\cos y,\ \partial_y f(x,y)=-e^x\sin y \Rightarrow \partial_x f(0,\pi)=-1,\ \partial_y f(0,\pi)=0\end{aligned}\right\}\Rightarrow$$

$\Rightarrow D_{\vec{u}}f(0,\pi)=-\frac{1}{\sqrt{2}}$ ■

· Gradient Vector

Def: If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\boxed{\nabla f(x,y)=\langle f_x(x,y),f_y(x,y)\rangle=\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}} \quad (3)$$

Note: $\boxed{D_{\vec{u}}f(x,y)=\nabla f(x,y)\cdot\vec{u}}$ ← This is just a reformulation of the equality (2)

Ex2: Find the gradient of $f = \frac{x^2}{y^3}$. Find the rate of change of f at point $P(3,1)$ in the direction of $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle \frac{2x}{y^3}, -\frac{3x^2}{y^4} \rangle$$

$$D_{\vec{u}} f(3,1) = \nabla f(3,1) \cdot \vec{u} = \langle 6, -27 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{18-108}{5} = -18$$

· Directional Derivatives and Gradient Vector for $f(x,y,z)$

All previous considerations for functions of two variables can be naturally generalized to the case of functions of 3 variables x, y, z.

Def: The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = \langle a, b, c \rangle$ is

$$\boxed{D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}} \quad (4)$$

Def: For a function of 3 variables, the gradient vector, denoted ∇f or grad f is

$$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}} \quad (5)$$

Similarly to functions of 2 variables, these two definitions are related via the following equality:

$$\boxed{D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}} \quad (6)$$

Ex3: Find the gradient of $f(x,y,z) = e^x \cos(yz)$.
Find the directional derivative of f at $P(1, \frac{1}{2}, \pi)$ in the direction of the vector $\vec{v} = 2\vec{i} - 2\vec{j} + \vec{k}$

$$\nabla f(x,y,z) = \langle e^x \cos(yz), -e^x \sin(yz) \cdot z, -e^x \sin(yz) \cdot y \rangle$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$$

$$D_{\vec{u}} f(1, \tfrac{1}{2}, \pi) = \langle 0, -e \cdot \pi, -\frac{e}{2} \rangle \cdot \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle = \frac{2e\pi}{3} - \frac{e}{6}$$

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- Max/min of $D_{\vec{u}} f$

Question: Given a function f and a point (x_0, y_0, z_0) in its domain what are the directions in which f increases/decreases with the maximal rate?

Recall that both for f's of 2 or 3 variables we have

$$D_{\vec{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \vec{u}$$

Here $\nabla f(x_0, y_0, z_0)$ is a fixed vector while $\vec{u}$ - unit vector which varies

$\Rightarrow \nabla f(x_0, y_0, z_0) \cdot \vec{u} = |\nabla f(x_0, y_0, z_0)| \cdot 1 \cdot \cos(\theta)$, where θ is the angle b/w $\nabla f(x_0, y_0, z_0)$ and $\vec{u}$.

Recall $-1 \leq \cos(\theta) \leq 1$, and $\cos(\theta) = 1 \Leftrightarrow \theta = 0 \pmod{2\pi}$
$\cos(\theta) = -1 \Leftrightarrow \theta = \pi \pmod{2\pi}$.

Upshot:
- The maximal value of the directional derivative $D_{\vec{u}} f(X)$ is $|\nabla f(X)|$ and it occurs when $\vec{u}$ has same direction as $\nabla f(X)$.
- The minimal value of the directional derivative $D_{\vec{u}} f(X)$ is $-|\nabla f(X)|$ and it occurs when $\vec{u}$ has direction opposite to that of $\nabla f(X)$.

Ex 4: Consider $f(x,y,z) = e^x \cos(yz)$ and $P = (1, \frac{1}{2}, \pi)$ as in Ex3
In what direction does f have the max/min directional deriv.
What are the corresponding values of these?

- Tangent Plane to Level Surfaces

PostPoned till next class

Given a function $F(x,y,z)$ one can consider the level surfaces of it (similarly to level curves for functions of two variables) as the surfaces given by the equation $F(x,y,z) = k$ for every number k in the range of F.

Want: Construct/define a tangent plane to this level surface at each pt.