- Last time

(1) Discussed problems on local min/max values of $f(x,y)$

- Compute $f_x(x,y)$, $f_y(x,y)$ to find critical points (where $f_x = f_y = 0$)
- Apply second derivative test to decide if each critical point is a local min, local max, saddle point (or in some cases the second derivative test will not tell exactly).

(2) Discussed problems on absolute min/max values of $f(x,y)$ in $D \subset \mathbb{R}^2$.

- Find critical points (choose only those which belong to D)
- Split boundary into several pieces, so that each of them can be simply parametrized by one variable, and find extreme values of the restriction of f to each such piece of a boundary (the latter is a function of 1 variable, so you should know how to handle them).
- Evaluate f at the critical points and extreme points on the boundary and choose the smallest and biggest values.

Ex 1: Find the point on the plane $x+y+5z-1=0$ that is closest to the point $(1,2,5)$

▸ The distance b/w $(1,2,5)$ and (x,y,z) is $\sqrt{(x-1)^2+(y-2)^2+(z-5)^2} =: d$.

Want: Find (x,y,z) on the plane, minimizing d, or equivalently minimizing d^2.

Note that $x+y+5z-1=0$ determines x uniquely given y,z: $x = 1-y-5z$.

Then $d^2 = \left(\sqrt{(-y-5z)^2+(y-2)^2+(z-5)^2}\right)^2 =: f(y,z)$. The minimal point is the critical point, i.e. $f_y = f_z = 0$.

$$f(y,z) = (y-2)^2+(z-5)^2+(y+5z)^2 \Rightarrow f_y(y,z) = 2(y-2)+2(y+5z) = 2(2y+5z-2)$$

$$f_z(y,z) = 2(z-5)+10(y+5z) = 10y+52z-10$$

Hence: $\begin{cases} 2y+5z-2=0 \\ 10y+52z-10=0 \end{cases} \Rightarrow \begin{cases} y=1 \\ z=0 \end{cases} \Rightarrow x = 1-1-0=0$

$f_{yy}(1,0)=4$, $f_{yz}(1,0)=10$, $f_{zz}(1,0)=52 \Rightarrow D = 4\cdot 52-10^2 = 110 > 0 \overset{f_{yy}(1,0)>0}{\Longrightarrow} (y,z)=(1,0)$ – local min.

It is quite clear that this local min is an absolute min. ■

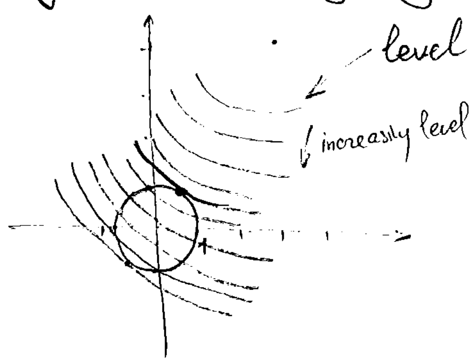
Lecture #10 10/03/2017

This Exercise is the simplest in the class of those, where you are asked to minimize or maximize a value of $f(x,y,z)$ given a constraint $g(x,y,z)=k$. In Ex1 above $f(x,y,z)=(x-1)^2+(y-2)^2+(z-5)^2$, while the constraint was $x+y+5z-1=0$. Note that we expressed one of the coordinates (x) via the other two (y,z) using this constraint and reduced the problem to the question of minimizing function on $\mathbb{R}^2$.

However: in most of the cases a constraint $g(x,y,z)=k$ does not allow to express one of the coordinates through the other two.

Ex2: Find a point on the unit circle $x^2+y^2=1$, which is most closely located to the point $(3,4)$.

Before we actually start computational part, we need to understand what is going on. We are minimizing $f(x,y)=(x-3)^2+(y-4)^2$ with a constraint $g(x,y)=1$, where $g(x,y)=x^2+y^2$.



Important/Key feature

At the min/max pt, the level curve of $f(x,y)$ is tangent to the graph of g. Equivalently their normal vectors are proportional! Recall that we can choose normal vectors to be ∇f, ∇g, resp.

$\nabla f(x,y)=\langle 2(x-3), 2(y-4)\rangle$
$\nabla g(x,y)=\langle 2x, 2y\rangle$

$$\Rightarrow \exists \lambda: \begin{cases} 2(x-3)=\lambda\cdot 2x \\ 2(y-4)=\lambda\cdot 2y \\ x^2+y^2=1 \end{cases} \Leftrightarrow \begin{cases} x-3=\lambda x \\ y-4=\lambda y \\ x^2+y^2=1 \end{cases} \Leftrightarrow \begin{cases} x=\frac{3}{1-\lambda} \\ y=\frac{4}{1-\lambda} \\ x^2+y^2=1 \end{cases}$$

So: $\left(\frac{3}{1-\lambda}\right)^2+\left(\frac{4}{1-\lambda}\right)^2=1 \Rightarrow \frac{25}{(1-\lambda)^2}=1 \Rightarrow 1-\lambda=\pm 5 \Rightarrow \left[\begin{array}{l}\lambda=-4\\ \lambda=6\end{array}\right.$

$\lambda=-4 \Rightarrow x=\frac{3}{5}, y=\frac{4}{5}$
$\lambda=6 \Rightarrow x=-\frac{3}{5}, y=-\frac{4}{5}$

$\Big\}$ hence $\left(\frac{3}{5},\frac{4}{5}\right)$ and $\left(-\frac{3}{5},-\frac{4}{5}\right)$ are the only two candidates for the closest pt.

As $f\left(\frac{3}{5},\frac{4}{5}\right) < f\left(-\frac{3}{5},-\frac{4}{5}\right)$ (we obviously see this via direct computations), we see that $(3/5, 4/5)$ is the closest point (while $(-3/5,-4/5)$ is the most distant one) ∎

Rmk: In Ex1 you could also apply the same reasoning, i.e. require $\langle 2(x-1), 2(y-2), 2(z-5)\rangle = \lambda\cdot\langle 1,1,5\rangle \Leftrightarrow x=1+\lambda/2,\ y=2+\lambda/2,\ z=5+5\lambda/2$, while we also use $x+y+5z-1=0$ to find λ and then recover back x,y,z.

General situation

In the previous Ex 2 we saw that when f, g depend on two variables, then the points of min/max of f under the given constraint $g(x,y)=k$ have a nice geometric property: level curves of f and the level curve given by $g(x,y)=k$ are tangent at this point. This is clear intuitively, but let us provide a reasonable math. argument for that and we'll consider a slightly more general case: when f, g depend on 3 variables.

Let (x_0, y_0, z_0) be the point on the surface S in $\mathbb{R}^3$ defined by $g(x,y,z)=k$. Then as argued last time (when we proved that the gradient ∇F is perpendicular to tangent vector of any curve on the level surface) consider the curve C on S given by vector equation $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ s.t. $x(t_0)=x_0$, $y(t_0)=y_0$, $z(t_0)=z_0$. Then the restriction of $f(\cdot,\cdot,\cdot)$ to this curve is a function $h(t) := f(x(t), y(t), z(t))$, which has a min/max at t_0.

Hence: $0 = h'(t_0) = f_x(x_0,y_0,z_0)\cdot x'(t_0) + f_y(x_0,y_0,z_0)\cdot y'(t_0) + f_z(x_0,y_0,z_0)\cdot z'(t_0)$
$$= \nabla f(x_0,y_0,z_0) \circ \vec{r}\,'(t_0).$$

Thus: $\vec{r}\,'(t_0)$ is perpendicular to $\nabla f(x_0,y_0,z_0)$ for every choice of C on S

But: as we saw last time $\vec{r}\,'(t_0)$ is also perpendicular to $\nabla g(x_0,y_0,z_0)$.

Therefore: If $\nabla g(x_0,y_0,z_0) \neq 0$, then there is a real number $\lambda \in \mathbb{R}$ s.t

$$\boxed{\nabla f(x_0,y_0,z_0) = \lambda \cdot \nabla g(x_0,y_0,z_0)}$$

Def: λ is called a Lagrange multiplier.

Method of Lagrange Multipliers

To find min/max values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$

(1) Find all values (x,y,z) and λ s.t.

$$\begin{cases} \nabla f(x,y,z) = \lambda \cdot \nabla g(x,y,z) \\ g(x,y,z)=k \end{cases}$$ ← this amounts to $f_x=\lambda\cdot g_x$, $f_y=\lambda\cdot g_y$, $f_z=\lambda\cdot g_z$

(2) Among the values of f at the points obtained in (1) choose the minimal and maximal. These are the minimal/maximal values of f.

Rmks : (1) We assume $\nabla g \neq 0$ on the surface $g(x,y,z)=k$

(2) We assume the absolute min/max exist.

(3) It is not necessary to find explicit values for λ.

(4) If f, g - functions of two variables, the same procedure applies.

Ex 3 : Find the min/max values of $f(x,y,z)=xyz$ on the sphere $x^2+y^2+z^2=12$

$\nabla f(x,y,z)=\langle yz, xz, xy\rangle$, $\nabla g(x,y,z)=\langle 2x, 2y, 2z\rangle$

$$\begin{cases} yz = 2\lambda x \\ xz = 2\lambda y \\ xy = 2\lambda z \\ x^2+y^2+z^2=12 \end{cases} \quad (*)$$

Multiplying the 1st eq-n by x, 2nd - by y, 3d - by z, we get

$$xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2. \quad (1)$$

Case 1 : $\lambda=0$

Then from the original system $(*)$, we find $yz=xz=xy=0$. Therefore, at least two of the coordinates are zero, while the third must be nonzero, due to the fourth equation of $(*)$. And clearly the value of f at these points is 0.

Case 2 : $\lambda \neq 0$

Then (1) implies $x^2=y^2=z^2$ and plugging this into the last eq-n of $(*)$, we get

$$x^2=y^2=z^2=4.$$

There are 8 points satisfying this condition : $(\pm 2, \pm 2, \pm 2)$ (with arbitrary choice of each of the three signs).

And it is clear that f is equal to ± 8 at each of these points.

Therefore : The maximum of f is 8, and it is achieved at

$(2,2,2)$, $(2,-2,-2)$, $(-2,2,-2)$, $(-2,-2,2)$.

The minimum of f is -8, and it is achieved at

$(-2,-2,-2)$, $(2,2,-2)$, $(2,-2,2)$, $(-2,2,2)$

Ex 4: Find the extreme values of $f(x,y) = e^{xy}$ on the region $D \subset \mathbb{R}^2$ given by $D = \{(x,y) \mid x^2+4y^2 \le 4\}$.

- In this problem, we combine the method from last lecture together with today's method. Indeed, first we find critical points of $f(x,y)$ in D.

(1) $f_x(x,y) = y \cdot e^{xy}$, $f_y(x,y) = x \cdot e^{xy}$ $\Rightarrow$ the only critical point is $(0,0)$ and it is inside D. The value of f at this point is $f(0,0) = e^0 = 1$.

(2) Next, we look for extreme values of $f(\cdot,\cdot)$ on the boundary of D. The latter is given by $\underbrace{x^2+4y^2}_{g(x,y)}=4$ and that is where we apply the method of Lagrange multipliers.

$$\begin{cases} y \cdot e^{xy} = \lambda \cdot 2x \\ x \cdot e^{xy} = \lambda \cdot 8y \qquad (*) \\ x^2+4y^2=4 \end{cases}$$

Multiplying the first eq-n by x, the second by y, we get $xye^{xy} = 2\lambda x^2 = 8\lambda y^2$.

Note that $\lambda \neq 0$. Indeed, if $\lambda = 0$, then $\left.\begin{array}{l} ye^{xy}=0 \Rightarrow y=0 \\ xe^{xy}=0 \Rightarrow x=0 \end{array}\right\}$ but $0^2+4 \cdot 0^2 \neq 4$

$\left.\begin{array}{l} \text{As } \lambda \neq 0 \text{, we get } 2\lambda x^2 = 8\lambda y^2 \Rightarrow x^2 = 4y^2 \\ \text{But the last eq-n of } (*) \text{ is } x^2+4y^2=4 \end{array}\right\} \Rightarrow x^2 = 4y^2 = 2$

Hence: $x = \pm\sqrt{2}$, $y = \pm\frac{1}{\sqrt{2}}$ and we get 4 points: $(\pm\sqrt{2}, \pm 1/\sqrt{2})$.

Let us now compute f at these points:

$f(\sqrt{2}, 1/\sqrt{2}) = e$, $f(\sqrt{2}, -1/\sqrt{2}) = \frac{1}{e}$, $f(-\sqrt{2}, 1/\sqrt{2}) = \frac{1}{e}$, $f(-\sqrt{2}, -1/\sqrt{2}) = e$

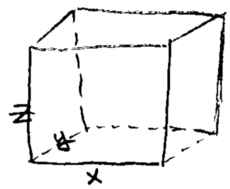
Note: $\frac{1}{e} < 1 < e$.

Therefore: The maximum of f is e, and it is achieved at $(\sqrt{2}, \frac{1}{\sqrt{2}})$, $(-\sqrt{2}, -1/\sqrt{2})$

The minimum of f is $\frac{1}{e}$, and it is achieved at $(\sqrt{2}, \frac{-1}{\sqrt{2}})$, $(-\sqrt{2}, \frac{1}{\sqrt{2}})$ ■

Lecture #10 10/03/2017

Ex 5: A box is to be constructed with a volume of 500 cubic inches. The box has 4 sides and a bottom, but no top. What are the dimensions of the cheapest box.



Volume $= xyz$

Surface Area $= xy + 2xz + 2yz$

Set $f(x,y,z) = xy + 2xz + 2yz$, $g(x,y,z) = xyz$.

Want to minimize $f(x,y,z)$, given the constraint $g(x,y,z) = 1372$.

$$\begin{cases} y + 2z = \lambda yz \\ x + 2z = \lambda xz \\ 2x + 2y = \lambda xy \\ xyz = 500 \end{cases} \quad (*)$$

Let us multiply the first three eq-s of $(*)$ by x, y, z, respectively (note that none of them is ZERO, due to the last equality of $(*)$).

Then: $\lambda xyz = xy + 2xz = xy + 2yz = 2xz + 2yz$.

- $xy + 2xz = xy + 2yz \Rightarrow z(x-y) = 0 \overset{z \neq 0}{\Rightarrow} x = y$
- $xy + 2yz = 2xz + 2yz \Rightarrow x(y - 2z) = 0 \overset{x \neq 0}{\Rightarrow} y = 2z$

So: $x = y = 2z \Rightarrow xyz = 4z^3$
But $xyz = 500$
$\Big\} \Rightarrow 4z^3 = 500 \Rightarrow z^3 = 125 \Rightarrow z = 5 \Rightarrow x = y = 10.$

Hence: There is only one triple (x,y,z) and $\lambda \in \mathbb{R}$ satisfying $(*)$ and the corresponding point is $(10, 10, 5)$.

Therefore, the dimensions of the cheapest box are $5 \times 10 \times 10$ inches ∎

Ex6*: (a) Maximize $f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$ subject to the constraint $x_1^2 + \ldots + x_n^2 = 1$

(b) Maximize $f(x_1, \ldots, x_n) = \sqrt[n]{x_1 \cdot \ldots \cdot x_n}$ subject to the constraint $x_1 + \ldots + x_n = 1$, together with $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$.

Ex 7: Find min/max values of $\ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1)$ given $x^2+y^2+z^2 = 12$