Lecture #11 10/05/2017

- Last time

∘ Last time we learnt the method of Lagrange multipliers, which is used to find min/max values of $f(x,y,\ldots)$ given one constraint $g(x,y,\ldots) = k$ ← constant.

Rmk: If the constraint is given by inequality, e.g. $g(x,y,\ldots) \leq k$, then we first find the critical points of $f$ satisfying this inequality, but then also find extreme values on the boundary (given by $g(x,y,\ldots)=k$) via Lagrange multipliers (or elementary tools as discussed last Thur if the boundary can be easily parametrized by 1 parameter).

- Compare the answers to the last Ex 5 from last time
- Ask if there are any questions before switching to a new topic

Exam

The First Midterm will be held today, October 5, at $7^{30}$ - $9^{00}$ pm in Davies.

Multiple Integrals

Goal: Extend the familiar notion of the definite integral of a function of 1 variable to the case of double and triple integrals of functions of 2 or 3 variables.

Warning: A precise definition of double and triple integrals is pretty similar to the case of usual $\int_a^b f(x)dx$ and involves Riemann sums. In this class, we are skipping this part. However, it is important to keep in mind that double integrals can be utilized to compute the "oriented" volume under $z=f(x,y)$ in the same way usual integrals are used to compute areas.

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Since it is hardly ever possible to evaluate the integral via a rigorous definition, we need some other tools

- **Iterated Integral**

Given a function $f(x,y)$ on the rectangle $[a,b]\times[c,d]$, we can evaluate two iterated integrals

$$\int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy\right)}_{\text{over here } x \text{ is fixed and we integrate w.r.t. } y} dx \quad \text{and} \quad \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx\right)}_{\text{over here } y \text{ is fixed and we integrate w.r.t. } x} dy$$

**Ex1**: Evaluate the iterated integrals

(a) $\int_0^1\int_0^2 xe^y\,dy\,dx$ $\left[\int_0^1\int_0^2 xe^y dy dx = \int_0^1 \left(x\cdot e^y\Big|_{y=0}^{y=2}\right)dx = (e^2-1)\cdot\int_0^1 x\,dx = \frac{e^2-1}{2}\right]$

(b) $\int_0^2\int_0^1 xe^y\,dx\,dy$ $\left[\int_0^2\int_0^1 xe^y dx dy = \int_0^2 \left(e^y\cdot\frac{x^2}{2}\Big|_{x=0}^{x=1}\right)dy = \frac{1}{2}\int_0^2 e^y dy = \frac{e^2-1}{2}\right]$

! Get the same answers

Fubini's Theorem: If $f(x,y)$ is continuous on the rectangle $R=[a,b]\times[c,d]$, i.e. $R=\{(x,y)\mid a\le x\le b,\ c\le y\le d\}$, then

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy}$$

**Ex2**: Evaluate $\iint_R 2ye^{xy}\,dA$, where $R=[0,1]\times[-1,1]$.

$\left[\iint_R 2ye^{xy}dA = \int_{-1}^1\int_0^1 2ye^{xy}dx\,dy = \int_{-1}^1\left(2e^{xy}\Big|_{x=0}^{x=1}\right)dy = \int_{-1}^1 2(e^y-1)dy = 2(e-e^{-1})-2\cdot2 = 2e-\frac{2}{e}-4\right]$

**Ex3**: Evaluate $\iint_R 6xy^2e^{x^2}\cos(y^3)\,dx\,dy$, where $R=[0,2]\times[0,1]$

Hint: Split it into $(2xe^{x^2})\cdot(3y^2\cos(y^3))$ $\left[= \int_0^2\left(6xe^{x^2}\int_0^1 y^2\cos(y^3)dy\right)dx = \int_0^2\left(6xe^{x^2}\cdot\frac{\sin(y^3)}{3}\Big|_{y=0}^{y=1}\right)dx\right.$
$\left.= \int_0^2 2\cdot\sin(1)\cdot xe^{x^2}dx = 2\sin(1)\cdot\frac{e^{x^2}}{2}\Big|_0^2 = (e^4-1)\cdot\sin(1)\right]$

**Important Application of Fubini's Theorem**

If $f(x,y)=g(x)\cdot h(y)$ on $R=[a,b]\times[c,d]$, then Fubini's theorem implies

$$\boxed{\iint_R g(x)h(y)\,dx\,dy = \int_a^b g(x)\,dx\cdot\int_c^d h(y)\,dy}$$

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**Ex 4**: Find the volume of the solid $S$ that is bounded by the elliptic paraboloid $4x^2+y^2+z=10$, the planes $x=1$, $y=2$ and the three coordinate planes. [ First, you need to note that $z=10-4x^2-y^2\geq 0$ in the region. Hence, $Vol=\int_0^1\int_0^2(10-4x^2-y^2)\,dy\,dx$, which is easy to compute... ]

**Ex 5**: Compute the following integrals:

(a) $\int_0^1\int_{-2}^1(y^2+y^3\sin x)\,dx\,dy$ $\left[=\int_0^1(y^2\cdot 3-y^3\cdot(\cos(1)-\cos(-2)))\,dy=y^3\Big|_{y=0}^{y=1}-\frac{\cos(1)-\cos(2)}{4}\,y^4\Big|_{y=0}^{y=1} = 1-\frac{\cos(1)-\cos(2)}{4}\right]$

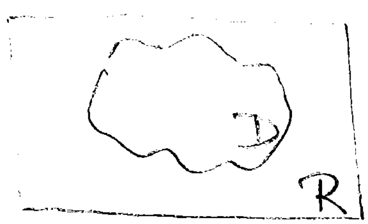
(b) $\iint_R x\cos(x+y)\,dA$, $R=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$ $\left[\int_0^{\pi/2}\int_0^{\pi/2}x\cos(x+y)\,dy\,dx=\int_0^{\pi/2}x\sin(x+y)\Big|_{y=0}^{y=\pi/2}dx = \int_0^{\pi/2}(x\cos x-x\sin x)\,dx=(x\sin x+x\cos x)\Big|_{x=0}^{x=\frac{\pi}{2}} - \int_0^{\pi/2}(\sin x+\cos x)\,dx=\frac{\pi}{2}-2\right]$

(c) $\iint_R(ye^{xy}+x\sin(xy))\,dA$, $R=[0,1]\times[0,2]$ $\left[\int_0^2\int_0^1 ye^{xy}\,dx\,dy+\int_0^1\int_0^2 x\sin(xy)\,dy\,dx = \int_0^2\left(e^{xy}\Big|_{x=0}^{x=1}\right)dy+\int_0^1(-\cos(xy))\Big|_{y=0}^{y=2}dx = \int_0^2(e^y-1)\,dy-\int_0^1(\cos(2x)-1)\,dx = e^2-1-2-\frac{\sin(2)}{2}+1 = e^2-\frac{\sin(2)}{2}-2\right]$

### • Double Integrals over General Regions

In the above discussions, we learnt how to integrate $f(x,y)$ over the rectangular region $R=[a,b]\times[c,d]$. However, in contrast to the 1-dimensional cases, there are way more regions to consider.

Idea: Find a big enough rectangle $R=[a,b]\times[c,d]$ containing our region $D$




Set

$$F(x,y):=\begin{cases} f(x,y), & (x,y)\in D\\ 0, & (x,y)\in R\setminus D\end{cases}$$

It is clear that the volume under $F(x,y)$ over $R$ is the same as the volume under $f(x,y)$ over $D$. For this reason, we define the double integral of $f$ over $D$.

$$\boxed{\iint_D f(x,y)\,dA=\iint_R F(x,y)\,dA}$$

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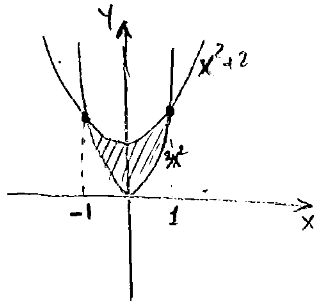
In practice, this amounts to computing iterated double integral, but with limits of integration being no longer fixed constants.

Ex 6: Evaluate $\iint_D (x-y)\,dA$, where $D$ is the region bounded by

(a) the parabolas $y=3x^2$ and $y=x^2+2$

(b) the graphs of $y=\sin x$, $y=\cos x$ and $x$-axis with $0\le x\le \pi/2$

(a)




Solve $x^2+2=3x^2$ to find $x$-coordinates of intersection pts

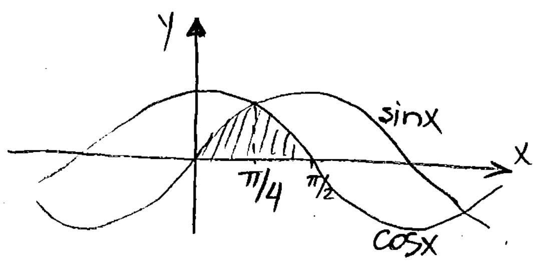
$x^2=1 \Leftrightarrow x=\pm 1$

$$\int_{-1}^{1}\int_{3x^2}^{x^2+2}(x-y)\,dy\,dx=\int_{-1}^{1}\left(x\cdot(x^2+2-3x^2)-\frac{y^2}{2}\Big|_{3x^2}^{y=x^2+2}\right)dx=$$

$$=\int_{-1}^{1}\left(-2x^3+2x-\frac{x^4+4x^2+4-9x^4}{2}\right)dx=\int_{-1}^{1}(4x^4-2x^3-2x^2+2x-2)\,dx=$$

$$=\left(\frac{4}{5}x^5-\frac{1}{2}x^4-\frac{2}{3}x^3+x^2-2x\right)\Big|_{x=-1}^{x=1}=\frac{8}{5}-\frac{4}{3}-4=-\frac{56}{15}$$

(b)




Key observation: On the interval $[0,\pi/2]$

$\sin x=\cos x \Leftrightarrow \tan x=1 \Leftrightarrow x=\pi/4$. Moreover,

$\cos x\ge \sin x\ge 0$ for $0\le x\le \pi/4$

$0\le \cos x\le \sin x$ for $\frac{\pi}{4}\le x\le \pi/2$

Thus: $$\iint_D (x-y)\,dA=\int_0^{\pi/4}\int_0^{\sin x}(x-y)\,dy\,dx+\int_{\pi/4}^{\pi/2}\int_0^{\cos x}(x-y)\,dy\,dx=$$

$$=\int_0^{\pi/4}\left(x\sin x-\frac{\sin^2 x}{2}\right)dx+\int_{\pi/4}^{\pi/2}\left(x\cos x-\frac{\cos^2 x}{2}\right)dx$$

(1) $\displaystyle\int_0^{\pi/4}x\sin x\,dx=-x\cos x\Big|_{x=0}^{x=\pi/4}+\int_0^{\pi/4}\cos x\,dx=-\frac{\pi}{4\sqrt{2}}+\sin x\Big|_{x=0}^{x=\pi/4}=\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)$

(2) $\displaystyle\int_{\pi/4}^{\pi/2}x\cos x\,dx=x\sin x\Big|_{x=\pi/4}^{x=\pi/2}-\int_{\pi/4}^{\pi/2}\sin x\,dx=\frac{\pi}{2}-\frac{\pi}{4\sqrt{2}}+\cos x\Big|_{\pi/4}^{\pi/2}=\frac{\pi}{2}-\frac{\pi}{4\sqrt{2}}-\frac{1}{\sqrt{2}}$

(3) $\displaystyle\int_0^{\pi/4}\frac{\sin^2 x}{2}\,dx=\int_0^{\pi/4}\frac{1-\cos(2x)}{4}\,dx=\frac{1}{4}\cdot\left(\frac{\pi}{4}-0\right)-\frac{\sin(2x)}{8}\Big|_{x=0}^{x=\pi/4}=\frac{\pi}{16}-\frac{1}{8}$

(4) $\displaystyle\int_{\pi/4}^{\pi/2}\frac{\cos^2 x}{2}\,dx=\int_{\pi/4}^{\pi/2}\frac{1+\cos(2x)}{4}\,dx=\frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)+\frac{\sin(2x)}{8}\Big|_{x=\pi/4}^{x=\pi/2}=\frac{\pi}{16}-\frac{1}{8}$

So: $$\iint_D (x-y)\,dA=\frac{\pi}{2}-\frac{\pi}{4\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{\pi}{16}+\frac{1}{8}+\frac{\pi}{2}-\frac{\pi}{4\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{\pi}{16}+\frac{1}{8}$$

$$=\pi\left(1-\frac{1}{2\sqrt{2}}-\frac{1}{8}\right)-\sqrt{2}+\frac{1}{4}$$

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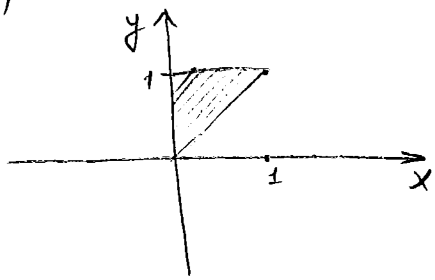
**Ex 7**: Evaluate the following integrals:

(a) $\iint_D \cos(2y^2)\,dA$ , $D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$

(b) $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$

(a)

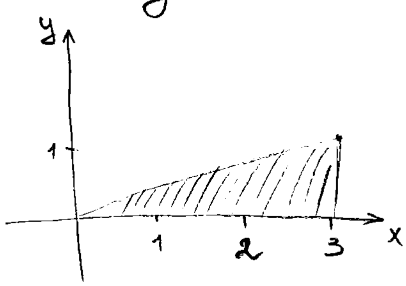


$$\iint_D \cos(2y^2)\,dA = \int_0^1 \int_0^y \cos(2y^2)\,dx\,dy = \int_0^1 \cos(2y^2)\cdot y \cdot dy$$

$$= \frac{\sin(2y^2)}{4}\Bigg|_{y=0}^{y=1} = \frac{\sin(2)}{4}$$

Warning: You could also write it as $\int_0^1 \int_x^1 \cos(2y^2)\,dy\,dx$, but here you will have trouble with computing inner integral!

(b) Let us first draw the corresponding region



You can not evaluate inner integral $\int_{3y}^3 e^{x^2}\,dx$.

However, let us reverse the order of integration:

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 \frac{x}{3} e^{x^2}\,dx = \frac{e^{x^2}}{6}\Bigg|_{x=0}^{x=3} = \frac{e^9 - 1}{6}$$