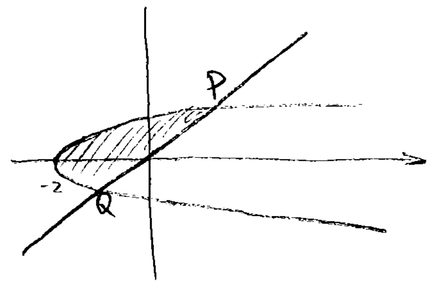
# Lecture #12

10/10/2017

- Let us do a few more examples on double integrals.

**Ex 1**: Evaluate $\iint_D y\,dA$, where $D$ is the region bounded by the line $x=y$ and the parabola $x=y^2-2$.

▶




Solve $\underbrace{y^2-2=y}$ to find $y$-coordinates of $P, Q$

$y^2-y-2=0 \Rightarrow \begin{cases} y=2 \\ y=-1 \end{cases}$

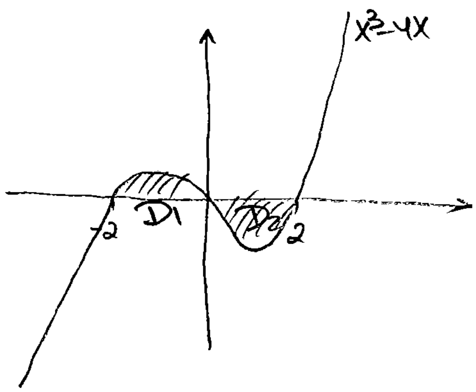
So: $\iint_D y\,dA = \int_{-1}^{2}\int_{y^2-2}^{y} y\,dx\,dy = \int_{-1}^{2} y(y-y^2+2)\,dy = \left(\frac{y^3}{3}-\frac{y^4}{4}+y^2\right)\Big|_{y=-1}^{y=2} =$

$= 3 - \frac{15}{4} + 3 = \frac{9}{4}$

■

**Ex 2**: Evaluate $\iint_D x^4\,dA$, where $D$ is bounded by $y=x^3-4x$ and $y=0$.

▶




$\iint_D x^4\,dA = \iint_{D_1} x^4\,dA + \iint_{D_2} x^4\,dA$

$\iint_{D_1} x^4\,dA = \int_{-2}^{0}\int_{0}^{x^3-4x} x^4\,dy\,dx = \int_{-2}^{0} x^4(x^3-4x)\,dx = \left(\frac{x^8}{8}-\frac{4}{6}x^6\right)\Big|_{x=-2}^{x=0}$

$= -32 + \frac{128}{3} = \frac{32}{3}$

$\iint_{D_2} x^4\,dA = \int_{0}^{2}\int_{x^3-4x}^{0} x^4\,dy\,dx = \int_{0}^{2} x^4(-x^3+4x)\,dx = \left(-\frac{x^8}{8}+\frac{4}{6}x^6\right)\Big|_{x=0}^{x=2} = \frac{32}{3}$

Answer: $\iint_D x^4\,dA = 64/3$.

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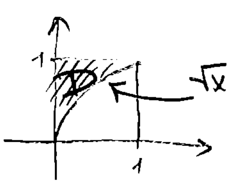
**Ex 3**: Evaluate $\int_0^1\int_{\sqrt{x}}^{1} \sqrt{y^3+1}\,dy\,dx$

▶ As you cannot compute the inner integral, let us rewrite in other order.




$\int_0^1\int_{\sqrt{x}}^{1} \sqrt{y^3+1}\,dy\,dx = \iint_D \sqrt{y^3+1}\,dA = \int_0^1\int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 y^2\sqrt{y^3+1}\,dy$

$\underset{u=y^3+1}{=} \int_1^2 \sqrt{u}\,\frac{du}{3} = \frac{2}{9}u^{3/2}\Big|_{u=1}^{u=2} = \frac{2}{9}(2\sqrt{2}-1)$

■

# Lecture #12 10/10/2017

• **Double integrals via polar coordinates**

If you have to integrate $f(x,y)$ over e.g. $D=\{(x,y) \mid x^2+y^2\le 1\}$, you have

$$\iint_D f(x,y)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

and due to the presence of expressions $\sqrt{1-\_\_^2}$ it may be very hard to compute the outer integral

Instead: Use polar coordinates $(r,\theta)$

Recall: $x=r\cos\theta$, $y=r\sin\theta$.

Analogously to the rectangle in xy-plane with sides parallel to axes, we have the notion of polar rectangle:

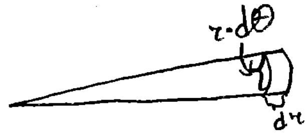
$$R=\{(r,\theta) \mid \underset{0}{\underset{\vee\!\!/}{a}}\le r\le b,\ \underbrace{\alpha\le\theta\le\beta}_{\beta-\alpha\le 2\pi}\}$$ – polar rectangle

Thm 1: If $f(\cdot,\cdot)$ is continuous on a polar rectangle $R=\{(r,\theta) \mid a\le r\le b,\ \alpha\le\theta\le\beta\}$, then

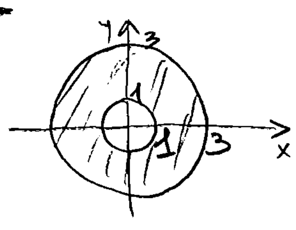
$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\cdot r\cdot dr\,d\theta}$$

Never forget this factor!

Hint for $dA \rightsquigarrow r\,dr\,d\theta$




Ex4: Evaluate $\iint_R 4(2x^2-y^2)\,dA$, where $R$ is bounded by $x^2+y^2=1$ and $x^2+y^2=9$




$$\iint_R 4(2x^2-y^2)\,dA = \int_0^{2\pi}\int_1^3 4\cdot(2r^2\cos^2\theta - r^2\sin^2\theta)\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}(160\cdot\cos^2\theta - 80\cdot\sin^2\theta)\,d\theta = \int_0^{2\pi}\left(160\cdot\frac{1+\cos(2\theta)}{2} - 80\cdot\frac{1-\cos(2\theta)}{2}\right)d\theta$$

$$= \boxed{80\pi}$$

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**Ex 5**: Find the volume of the solid bounded by $z=0$ and $z=4-x^2-y^2$.

$$Vol = \iint_D |4-x^2-y^2| dA = \int_0^{2\pi}\int_0^2 (4-r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \left(2r^2 - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=2} d\theta =$$

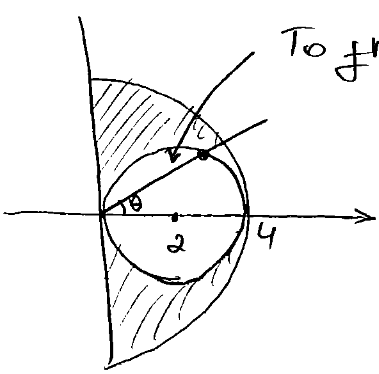
$$= 2\pi \cdot 4 = 8\pi$$

**Thm 2**: If $f(\cdot,\cdot)$ is continuous on the polar region of the form $D = \{(r,\theta) \mid \underbrace{\alpha \le \theta \le \beta}_{\beta-\alpha\le 2\pi},\ \underset{\overset{\vee}{0}}{h_1(\theta)} \le r \le h_2(\theta)\}$, then

$$\boxed{\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\, dr\, d\theta}$$

**Ex6**: Compute $\iint_R y\, dA$, where $R$ is the region bounded by $x^2+y^2=16$ and $x^2+y^2=4x$ in the half-plane $x\ge 0$.

To draw the region, note $x^2+y^2=4x \overset{\text{complete squares}}{\Longleftrightarrow} (x-2)^2+y^2=4$.




To find distance from the origin to the intersection point solve $r^2 = 4r\cos\theta \overset{r>0}{\Longrightarrow} r = 4\cos\theta$

$$\iint_R y\, dA = \int_{-\pi/2}^{\pi/2}\int_{4\cos\theta}^{4} r\sin\theta \cdot r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \sin\theta \cdot \frac{r^3}{3}\Bigg|_{r=4\cos\theta}^{r=4} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{81}{3}\sin\theta - \frac{81}{3}\sin\theta\cos^3\theta\right) d\theta$$

$$\int_{-\pi/2}^{\pi/2} \sin\theta\, d\theta = -\cos\theta \Big|_{\theta=-\pi/2}^{\theta=\pi/2} = 0$$

$$\int_{-\pi/2}^{\pi/2} \sin\theta\cos^3\theta\, d\theta = -\int_{-\pi/2}^{\pi/2} \cos^3\theta\, d(\cos\theta) = -\frac{\cos^4\theta}{4}\Bigg|_{\theta=-\pi/2}^{\theta=\pi/2} = 0$$

$$\Rightarrow \iint_R y\, dA = 0.$$