Lecture #13

10/12/2017

- Vector fields

Let D be a set in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function F that assigns to each point $(x,y)\in D$ a two-dimensional vector $F(x,y)$.

$F(x,y) = \langle P(x,y), Q(x,y)\rangle$

scalar fields

Completely analogously, we have a similar definition in $\mathbb{R}^3$:

Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function F that assigns to each point $(x,y,z)\in E$ a 3-dimensional vector $F(x,y,z)$.

$F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

Ex1: Describe the following vector fields by (1) providing a table of values at some points, (2) sketching these vectors, (3) explaining in words (if possible)

(a) $F(x,y) = y\vec{i} - x\vec{j}$

(b) $F(x,y,z) = y\cdot\vec{i}$

(c) $F(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$

Recall the notion of gradient fields:

$f(x,y) \rightsquigarrow \nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$

$f(x,y,z) \rightsquigarrow \nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$

Def: A vector field F is called a conservative vector field if it is the gradient of some scalar function, i.e. $\exists f$ such that $F=\nabla f$.

Def: In the above setting f – potential of F

Ex2: Draw gradient vector field of $f(x,y)=xy$.

Ex 3: Find the gradient vector fields of:

(a) $f(x,y) = \frac{1}{2}(x^2+y^2+z^2)$ [Deduce Ex 1(c) is conservative]

(b) $f(x,y) = e^x \sin(2xy)$

! It is important to keep in mind that gradient vector fields are always perpendicular to level curves.

Ex 4: Describe/Sketch the gradient vector field for $f(x,y) = e^{2x+4y}$

• Line integrals

Let us be given a curve C parametrized as $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$

Recall: Length $(C) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Line Integral of f along C: $\boxed{\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$

Note: Since we didn't give a very rigorous definition of the LHS (discussed it briefly in the class, this result can be taken as a definition)

Rmk: This definition is independent of the parametrization of C.

Completely similarly, for a smooth space curve C given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$, and $f(x,y,z)$ - continuous, then define

$$\boxed{\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

Ex 5: Evaluate $\int_C (5+xy^2)\, ds$, where C is the unit circle.

► $C = \{(\cos t, \sin t) \mid 0 \le t \le 2\pi\}$. Note $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1$

$$\int_C (5+xy^2)\, ds = \int_0^{2\pi} (5 + \cos t \sin^2 t) \cdot 1 \cdot dt = 10\pi + \int_0^{2\pi} \sin^2(t)\, d(\sin(t)) = 10\pi$$

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Ex6: Evaluate :

(a) $\int_C 3y\,ds$, $C = \{(t^2, 2t) \mid 0 \le t \le 3\}$

(b) $\int_C xe^y ds$, C – line segment from (1,1) to (4,5)

(c) $\int_C xe^{2yz} ds$, C – line segment from (0,0,0) to (1,3,2)

(a) $\int_C 3y\,ds = \int_0^3 3\cdot 2t\cdot\sqrt{4t^2+4}\,dt = \int_0^3 12t\sqrt{t^2+1}\,dt \overset{u=t^2+1}{=\!=} \int_1^{10} \sqrt{u}\cdot 6du = 6\cdot\frac{2}{3}u^{3/2}\Big|_{u=1}^{u=10}$

$= \boxed{4(10\sqrt{10}-1)}$

(b) $C = \{(1+3t, 1+4t) \mid 0 \le t \le 1\}$

$\int_C xe^y ds = \int_0^1 (1+3t)e^{1+4t}\cdot\sqrt{3^2+4^2}\,dt = 5\int_0^1 (1+3t)e^{1+4t}dt \overset{u=1+4t}{=\!=}$

$= 5\int_1^5 \left(\frac{3}{4}u+\frac{1}{4}\right)e^u\cdot\frac{du}{4} = \frac{15}{16}\int_1^5 ue^u du + \frac{5}{16}\int_1^5 e^u du$

$= \frac{15}{16}\left((ue^u)\Big|_{u=1}^{u=5} - \int_1^5 e^u du\right) + \frac{5}{16}\int_1^5 e^u du = \boxed{\frac{15}{16}(5e^5-e) - \frac{10}{16}(e^5-e)}$

always simplify further...

(c) $C = \{(t, 3t, 2t) \mid 0 \le t \le 1\}$

$\int_C xe^{2yz} ds = \int_0^1 t\cdot e^{2\cdot 3t\cdot 2t}\cdot\sqrt{1^2+3^2+2^2}\,dt = \sqrt{14}\cdot\int_0^1 te^{12t^2}dt$

$\overset{u=t^2}{=\!=} \sqrt{14}\cdot\int_0^{12} e^u du\cdot\frac{1}{2} = \boxed{\frac{\sqrt{14}}{2}(e^{12}-1)}$

Def: Line integrals of $f(x,y)$ along C with respect to x or y are:

$$\boxed{\begin{aligned}\int_C f(x,y)\,dx &= \int_a^b f(x(t),y(t))\cdot x'(t)\,dt\\ \int_C f(x,y)\,dy &= \int_a^b f(x(t),y(t))\cdot y'(t)\,dt\end{aligned}}$$

In 3-dim case you will also have "... with respect to z".