Lecture #14 10/17/2017

- Last time

* Last time we learnt a concept of vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$.
* An important class of vector fields (which will come up in the next few lectures) is a class of conservative vector fields: $F = \nabla f$ for some f-n f.
* Notational Conventions: F – vector field
  f – function

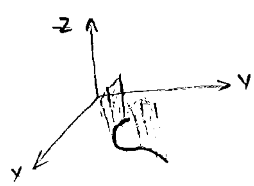
* Line Integral of f along C is computed as follows:

Plane Curve C: $\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Space Curve C: $\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

* Meaning of line integrals: (1) if you have a rope stretched out along the curve C with density function non-constant, then the line integral of density function along the curve is the mass of the rope

(2) Geometrically, if $f(x,y) \geq 0$, then $\int_C f(x,y)$ equals the area of the "fence" above the curve.



- Finally, we had a notion of line integrals of f along C w.r.t. x, or y, or z:

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t)) \cdot x'(t)\,dt$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t), y(t), z(t)) \cdot y'(t)\,dt$$

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t), y(t), z(t)) \cdot z'(t)\,dt$$

Notations: $\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$ means a sum of 3 integrals as above.

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Ex 1: Evaluate $\int_C x\,dx + y\,dy$ along the following curves:

(a) C is a line segment from $(-1,0)$ to $(1,0)$

(b) C is a half of the unit circle going clockwise from $(-1,0)$ to $(1,0)$

(c) C is part of the parabola $y = x^2 - 1$ with $-1 \le x \le 1$ (going from $(-1,0)$ to $(1,0)$)

(a) $C = \{(t,0) \mid -1 \le t \le 1\} \Rightarrow \int_C x\,dx + y\,dy = \int_{-1}^{1} t\,dt = 0$.

(b) $C = \{(\cos t, \sin t) \mid t$ ranging from π to $0\} \Rightarrow$

$$\Rightarrow \int_C x\,dx + y\,dy = \int_{\pi}^{0} \cos t \cdot (-\sin t)\,dt + \sin t \cdot \cos t \cdot dt = 0$$

(c) $C = \{(t, t^2-1) \mid t$ ranging from -1 to $1\} \Rightarrow \int_C x\,dx + y\,dy = \int_{-1}^{1} t\,dt + (t^2-1)\cdot 2t\,dt =$

$$= \int_{-1}^{1} (2t^3 - t)\,dt = \left(\frac{t^4}{2} - \frac{t^2}{2}\right)\Big|_{-1}^{1} = 0.$$

In particular, we see that the answer was the same for 3 different curves with the same endpoints.

But: this is not always the case

Ex 2: Evaluate $\int_C e^x dx + xy\,dy$ for the following curves:

(a) C is a part of parabola $y = x^2$ starting from $(0,0)$ ending at $(2,4)$.

(b) C is a line segment from $(0,0)$ to $(2,4)$.

(c) C consists of a line segment from $(0,0)$ to $(1,1)$, followed up by the line segment from $(1,1)$ to $(2,4)$

(a) $C = \{(t,t^2) \mid 0 \le t \le 2\}$: $\int_C e^x dx + xy\,dy = \int_0^2 (e^t + t \cdot t^2 \cdot 2t)\,dt = \left(e^t + \frac{2}{5}t^5\right)\Big|_0^2 = e^2 - 1 + \frac{64}{5}$

(b) $C = \{(2t, 4t) \mid 0 \le t \le 1\}$: $\int_C e^x dx + xy\,dy = \int_0^1 e^{2t} \cdot 2 \cdot dt + 2t \cdot 4t \cdot 4\,dt = e^{2t}\Big|_{t=0}^{t=1} + \frac{32}{3}t^3\Big|_{t=0}^{t=1}$

$$= e^2 - 1 + \frac{32}{3}$$

(c) $C = C_1 \sqcup C_2$, $C_1 = \{(t,t) \mid 0 \le t \le 1\}$, $C_2 = \{(1+t, 1+3t) \mid 0 \le t \le 1\}$

$$\int_{C_1} e^x dx + xy\,dy = \int_0^1 e^t dt + t \cdot t \cdot dt = \left(e^t + \frac{t^3}{3}\right)\Big|_{t=0}^{t=1}$$

$$\int_{C_2} e^x dx + xy\,dy = \int_0^1 e^{1+t} dt + (1+t)(1+3t)\cdot 3\,dt = \int_1^2 e^u du + 3\int_0^1 (1+4t+3t^2)\,dt = e^u\Big|_{u=1}^{u=2} + 3(t^3+2t^2+t)\Big|_{t=0}^{t=1}$$

$$\int_C e^x dx + xy\,dy = \int_{C_1} -''- + \int_{C_2} -''- = e^2 - 1 + \frac{1}{3} + 3(1+2+1)$$

In particular, in this example, we see that the value of integral of f along the curve C joining two points DOES depend on the curve.

Rmk: The reason why in Ex1 it didn't depend on the curve was due to the fact that $\langle x, y\rangle$ is a conservative vector field. We will come to this latter on.

Rmk: Note that while $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$, where $-C$ denotes the same curve C, but passed in the opposite direction, we have $\int_C f(x,y)\,dx = -\int_{-C} f(x,y)\,dx$, $\int_C f(x,y)\,dy = -\int_{-C} f(x,y)\,dy$ etc.

Ex3: Evaluate $\int_C xye^{yz}\,dy$, $C: x=t, y=t^2, z=t^3, 0 \le t \le 1$.

$$\int_0^1 t \cdot t^2 e^{t^5} \cdot 2t\,dt = \int_0^1 2t^4 e^{t^5}\,dt = \frac{2}{5} e^{t^5}\Big|_{t=0}^{t=1} = \frac{2}{5}(e-1)$$

∎

- Line integrals of vector fields

Def: Let F be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$. The line integral of F along C is

$$\boxed{\int_C F\,d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}$$

this also equals $\int_C F \cdot \vec{T}\,ds$, where $\vec{T}$ - unit tangent vector to C at a given point.

Ex 4: Evaluate $\int_C F\,d\vec{r}$, where $F(x,y,z) = x\vec{i} + y^2\vec{j} + z^3\vec{k}$, while C is given by $x=t, y=t^2, z=t^3$, $0 \le t \le 1$.

$$\int_C F\,d\vec{r} = \int_0^1 (t\vec{i} + t^4\vec{j} + t^9\vec{k}) \cdot (\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt = \int_0^1 (t + 2t^5 + 3t^{11})\,dt = \left(\frac{t^2}{2} + \frac{t^6}{3} + \frac{t^{12}}{4}\right)\Big|_{t=0}^{t=1}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

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Let us give a connection to the previous discussions $\left(\begin{matrix} F = P\vec{i} + Q\vec{j} + R\vec{k} \\ C: \vec{r}(t),\ a \le t \le b \end{matrix} \right)$

$$\int_C F\,dr = \int_a^b (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot (x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k})\,dt$$

$$= \int_a^b \left[P(x(t), y(t), z(t))\,x'(t) + Q(x(t), y(t), z(t))\,y'(t) + R(x(t), y(t), z(t))\,z'(t)\right]dt$$

$$= \int_a^b P\,dx + Q\,dy + R\,dz$$

So: $\boxed{\int_C F\,dr = \int_C P\,dx + Q\,dy + R\,dz}$

Ex 5: Evaluate the line integral $\int_C F\,dr$, where

$F(x,y) = e^x\,\vec{i} - \sin(y)\,\vec{j}$, C is given by $\vec{r}(t) = t^2\vec{i} + t^3\vec{j}$, $0 \le t \le 1$

– $\left.\begin{matrix} F(x,y) = \langle e^x, -\sin y\rangle \\ \vec{r}(t) = \langle t^2, t^3\rangle \end{matrix}\right\} \Rightarrow \begin{matrix} F(\vec{r}(t)) = \langle e^{t^2}, -\sin(t^3)\rangle \\ \vec{r}\,'(t) = \langle 2t, 3t^2\rangle \end{matrix}$

Hence, $\int_C F\,dr = \int_0^1 \langle e^{t^2}, -\sin(t^3)\rangle \cdot \langle 2t, 3t^2\rangle\,dt = \int_0^1 2te^{t^2}\,dt - \int_0^1 3t^2\sin(t^3)\,dt$

$$= e^u\Big|_{u=0}^{u=1} + \cos(u)\Big|_{u=0}^{u=1} = e + \cos(1) - 2$$ ∎

Ex 6: Evaluate $\int_C F\,dr$, where $F(x,y) = xy^2\,\vec{i} - x^3\,\vec{j}$

$C: \vec{r}(t) = t^3\vec{i} + t^4\vec{j}$, $0 \le t \le 1$

– $\left.\begin{matrix} F(\vec{r}(t)) = \langle t^{11}, -t^9\rangle \\ \vec{r}\,'(t) = \langle 3t^2, 4t^3\rangle \end{matrix}\right\} \Rightarrow \int_C F\,dr = \int_0^1 (3t^{13} - 4t^{12})\,dt = \left(\frac{3}{14}t^{14} - \frac{4}{13}t^{13}\right)\Big|_{t=0}^{t=1} = \frac{3}{14} - \frac{4}{13}$ ∎

Ex 7: Evaluate $\int_C F\,dr$, where $F(x,y,z) = \sin x \cdot \vec{i} + \cos y \cdot \vec{j} + xz\,\vec{k}$

$C: \vec{r}(t) = t^3\vec{i} - t^2\vec{j} + t\vec{k}$, $0 \le t \le 1$.

– $\left.\begin{matrix} F(\vec{r}(t)) = \langle \sin(t^3), \cos(-t^2), t^4\rangle \\ \vec{r}\,'(t) = \langle 3t^2, -2t, 1\rangle \end{matrix}\right\} \Rightarrow \int_C F\,dr = \int_0^1 (3t^2\sin(t^3) - 2t\cos(t^2) + t^4)\,dt$

$$= -\cos u\Big|_{u=0}^{u=1} - \sin u\Big|_{u=0}^{u=1} + \frac{1}{5}t^5\Big|_{t=0}^{t=1}$$

$$= \frac{6}{5} - \sin(1) - \cos(1)$$ ∎