Lecture #15 10/24/2017

- Fundamental Theorem for Line Integrals

Recall: If $F'(x)$ is a continuous function on $[a,b]$, then Fund. Thm of Calculus asserts $\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$

This result admits a natural generalization for smooth plane/space curves

Theorem 1: Let C be a smooth curve given by the position vector function $\vec{r}(t)$, $a \le t \le b$; let f be a differentiable function of two or three variables whose gradient vector field ∇f is continuous on C. Then:

$$\boxed{\int_C \nabla f\,dr = f(\vec{r}(b)) - f(\vec{r}(a))}$$

Proof of thm 1 (space curve)

$$\int_C \nabla f\,dr = \int_a^b \left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\right)dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt \overset{\text{Fundamental Theorem of Calculus}}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$ ∎

Independence of Path

Last time we saw that generally speaking, picking two different piecewise smooth curves C_1, C_2 starting from A and terminating at point B, we get

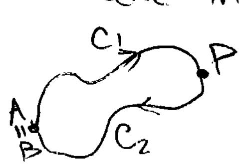
$$\boxed{\int_{C_1} F\,dr \neq \int_{C_2} F\,dr}$$

However, one implication of Thm 1 is that if ∇f - continuous, then:

$$\boxed{\int_{C_1} \nabla f\,dr = \int_{C_2} \nabla f\,dr}$$

i.e. the line integral of a conservative vector field is independent of path.

Let us now look at vector fields F s.t. $\int_C F\,dr$ is independent of path. For every closed curve C (i.e. points A & B coincide), take a point P somewhere in between. Then $\int_C F\,dr = \int_{C_1} F\,dr + \int_{C_2} F\,dr = \int_{C_1} F\,dr - \int_{-C_2} F\,dr \Rightarrow \int_C F\,dr = 0$



Here $C_1, -C_2$ - two paths with the same starting and terminating points

Vice-versa, if $\int_C F\,dr = 0$ for every closed path, then $\int_C F\,dr$ is path independent

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Theorem 2: $\int_C F\,dr$ is independent of path in D if and only if $\int_C F\,dr = 0$ for every closed path C in D.

As mentioned above, every gradient vector field continuous on the region D is path independent (with paths inside D). The reverse is also almost true.

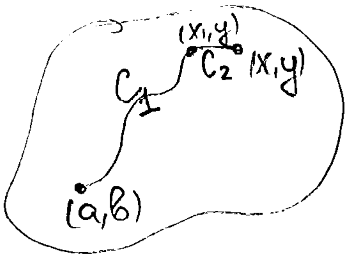
Theorem 3: Let D be open and connected. Suppose F is a vector field continuous on D. If $\int_C F\,dr$ is independent of path in D, then F is a conservative vector field on D, i.e. $F = \nabla f$ for some function f.

Proof of Theorem 3 (case of a plane curve)

- Fix a "starting" point $P(a,b) \in D$ and define a function

$$\boxed{f(x,y) := \int_{(a,b)}^{(x,y)} F\,dr} \qquad \text{for any } (x,y) \in D$$

Since $\int_C F\,dr$ is independent of path, it does not matter which path we take.



Choose $(x_1,y) \in D$ with $x_1 < x$ (such point exists as D open) and consider any path C_1 from (a,b) to (x_1,y) followed up by a horizontal segment C_2 from (x_1,y) to (x,y)

Then: $\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left(\int_{C_1} F\,dr + \int_{C_2} F\,dr\right) = \frac{\partial}{\partial x}\int_{C_2} F\,dr$

If $F = \langle P, Q\rangle$, then $\int_{C_2} F\,dr = \int_{x_1}^{x} P(t,y)\,dt \Rightarrow \frac{\partial}{\partial x}\int_{x_1}^{x} P(t,y)\,dt \overset{\text{Fundamental Theorem of Calculus}}{=} P(x,y)$ $\Rightarrow$

$\Rightarrow \frac{\partial}{\partial x} f(x,y) = P(x,y)$.

Analogously: $\frac{\partial}{\partial y} f(x,y) = Q(x,y)$

$\Rightarrow F = \langle P, Q\rangle = \nabla f$. $\blacksquare$

But still the question is given a vector field F how do we know it is conservative or not.

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One direction is easy and was already mentioned last time.

Theorem 4: If $F(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a conservative vector field, where P, Q have continuous first-order partial derivatives on a domain D, then we have

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } D}$$

The converse is not true in general, but is true under an additional assumption.

Def: The region $D \subset \mathbb{R}^2$ is simply-connected if every simple closed curve in D (i.e. without self-intersections) encloses only points that are in D.

Ex:  - simply-connected,  - not simply-connected

Theorem 5: Let $F = \langle P, Q\rangle$ be a vector field on an open simply-connected region D. Suppose P, Q have continuous first-order partial derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in D. Then, F is conservative.

Rmk: The proof is postponed till next lecture.

Note that Theorem 5 asserts an existence of f s.t. $F = \nabla f$, but it does not say how to find it. In the meanwhile, proof of Theorem 3 explains how to actually find f:

$$\boxed{f(x,y) = \int_{(a,b)}^{(x,y)} F\,d\vec{r}}$$

Algebraically, that means that knowing $f_x = P$, $f_y = Q$, you recover f up to a function in y by integrating P along x, while you recover that function in y by further integrating Q along y.

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Ex 1: Determine whether or not F is a conservative vector field. If it is, find potential f such that $F=\nabla f$.

(a) $F(x,y)=\langle 2xy+y^2, x^2+xy\rangle$

(b) $F(x,y)=\langle 2xy+y^2, x^2+2xy\rangle$

(c) $F(x,y)=\langle ye^x+\sin y, e^x+x\cos y\rangle$

(a) $\frac{\partial}{\partial y}(2xy+y^2)=2x+2y$
$\frac{\partial}{\partial x}(x^2+xy)=2x+y$ $\neq$ $\Rightarrow$ not conservative

(b) $\frac{\partial}{\partial y}(2xy+y^2)=2x+2y$
$\frac{\partial}{\partial x}(x^2+2xy)=2x+2y$ $\Rightarrow$ conservative.

To find f such that $f_x=2xy+y^2$, $f_y=x^2+2xy$, we first integrate f_x to find $f(x,y)$ up to a function in y

for every y: $\int(2xy+y^2)dx = x^2y+xy^2+\underset{\text{constant}}{C}$ $\Rightarrow$ $\boxed{f(x,y)=x^2y+xy^2+g(y)}$

To find $g(y)$, we differentiate previous equality:
$x^2+2xy=f_y=x^2+2xy+g'(y) \Rightarrow g'(y)=0 \Rightarrow g(y)=K$ – constant.

So: $\boxed{f(x,y)=x^2y+xy^2+K}$

(c) $\frac{\partial}{\partial y}(ye^x+\sin y)=e^x+\cos y$
$\frac{\partial}{\partial x}(e^x+x\cos y)=e^x+\cos y$ $\Rightarrow$ conservative.

Step 1 (Find f up to a f-n in y)
$\int(ye^x+\sin y)dx = ye^x+x\sin y+C \Rightarrow \boxed{f(x,y)=ye^x+x\sin y+g(y)}$

Step 2 (Find g): $e^x+x\cos y=f_y(x,y)=e^x+x\cos y+g'(y) \Rightarrow g'(y)=0 \Rightarrow g(y)=K$

So: $\boxed{f(x,y)=ye^x+x\sin y+K}$

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Ex2: Find function f such that $F = \nabla f$ and use it to evaluate $\int_C F dr$ along the given curve C.

(a) $F(x,y) = \langle x^3y^4, x^4y^3 \rangle$

$C: \vec{r}(t) = \langle e^t \cos t, \sin^3 t \rangle, \quad 0 \le t \le 1$

(b) $F(x,y,z) = \langle yz e^{xy}, xz e^{xy}, e^{xy}+1 \rangle$

$C: \vec{r}(t) = \langle t, t^2, t^3 \rangle, \quad 0 \le t \le 1$

(a) First, we note that F is conservative as $\frac{\partial}{\partial y}(x^3y^4) = 4x^3y^3$, $\frac{\partial}{\partial x}(x^4y^3) = 4x^3y^3$

- To find potential, we integrate $\int x^3y^4 dx = \frac{1}{4}x^4y^4 + \underbrace{C}_{\text{depending on } y} \Rightarrow$

$\Rightarrow f(x,y) = \frac{x^4y^4}{4} + g(y)$.

We determine $g(y)$ by using $\left.\begin{array}{l}\frac{\partial}{\partial y} f(x,y) = x^4y^3 \\ \quad \| \\ x^4y^3 + g'(y)\end{array}\right\} \Rightarrow g'(y) = 0 \Rightarrow g(y) = \underset{\text{const}}{K}$

So: $\boxed{f(x,y) = \frac{x^4y^4}{4} + K}$ – general f-la of a potential

- To compute integral, we write $\int_C F dr = \int_C \nabla f dr$, where f is any potential, e.g. $f(x,y) = \frac{x^4y^4}{4}$ and apply Theorem 1:

$\int_C \nabla f dr = f(\vec{r}(1)) - f(\vec{r}(0)) = f(e\cos 1, \sin^3 1) - f(1,0) =$

$= \frac{1}{4}\left(e \cdot \cos(1) \cdot \sin^3(1)\right)^4$

Upshot: $\boxed{\int_C F dr = \frac{1}{4}\left(e \cdot \cos(1) \cdot \sin^3(1)\right)^4}$

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(b) • As we have a vector field on $\mathbb{R}^3$, we need to check three equalities for $F = \langle P, Q, R \rangle$:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

In our case $P = yze^{xy}$, $Q = xze^{xy}$, $R = e^{xy} + 1$

• $\frac{\partial P}{\partial y} = ze^{xy} + xyze^{xy} = \frac{\partial Q}{\partial x}$ ✓

• $\frac{\partial P}{\partial z} = ye^{xy} = \frac{\partial R}{\partial x}$ ✓

• $\frac{\partial Q}{\partial z} = xe^{xy} = \frac{\partial R}{\partial y}$ ✓

So: F is a conservative vector field.

• Next, we find potential by a similar procedure, but in 3 steps

Step 1: $\int yze^{xy}\,dx = ze^{xy} + C \Rightarrow f(x,y,z) = ze^{xy} + g(y,z)$

Step 2: $\left.\begin{array}{l} \frac{\partial}{\partial y} f(x,y,z) = xze^{xy} \\ \quad \| \\ xze^{xy} + \frac{\partial}{\partial y} g(y,z) \end{array}\right\} \Rightarrow \frac{\partial}{\partial y} g(y,z) = 0$

Integrating $\int dy$, we see $g(y,z) = h(z)$, i.e. $f(x,y,z) = ze^{xy} + h(z)$

Step 3: $\left.\begin{array}{l} \frac{\partial}{\partial z} f(x,y,z) = e^{xy} + 1 \\ \quad \| \\ e^{xy} + h'(z) \end{array}\right\} \Rightarrow h'(z) = 1 \Rightarrow h(z) = z + K$

So: Potentials: $\boxed{f(x,y,z) = ze^{xy} + z + K}$

Finally, we apply Theorem 1 to compute the integral

$$\int_C F\,dr = \int_C \nabla f\,dr = f(\vec{r}(1)) - f(\vec{r}(0)) \quad \text{for } f(x,y,z) = ze^{xy} + z.$$

$\vec{r}(0) = \langle 0,0,0 \rangle \Rightarrow f(\vec{r}(0)) = 0$

$\vec{r}(1) = \langle 1,1,1 \rangle \Rightarrow f(\vec{r}(1)) = e + 1$

So: $\boxed{\int_C F\,dr = e + 1}$