Lecture #16 10/26/2017

• Continuation of Tuesday's Topic

Ex1: Show that if the vector field $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.} \quad (*)$$

► If $F = \nabla f$, then $P = f_x$, $Q = f_y$, $R = f_z$.

The above 3 equalities follow from

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}.$$

Ex2: Show that the line integral $\int_C x\,dx + yz\,dy + e^y z^2\,dz$ is not independent of path.

► From Tuesday class, we know that if it was path independent, then $\langle x, yz, e^y z^2 \rangle$ would be conservative. The latter would imply validity of (*) from Ex1, but it is easy to check

$$\frac{\partial}{\partial z}(yz) = y \neq e^y z^2 = \frac{\partial}{\partial y}(e^y z^2).$$

Ex3: Find a function f such that $F = \nabla f$ and use it to evaluate $\int_C F\,d\vec{r}$ along the given curve C.

$$F = \langle \sin y, x\cos y + e^z, ye^z \rangle, \quad C: \vec{r}(t) = \langle 2t^{10} - t^9, t^{20} - t^{15} + t^{10}, 3t^{100} - 2t \rangle$$
$$0 \le t \le 1$$

► • You start by integrating $\int \sin y\,dx = x\sin y + \text{const} \Rightarrow f(x,y,z) = x\sin y + g(y,z)$

• Then, we substitute $\left.\begin{array}{l} f_y = x\cos y + e^z \\ \quad\parallel \\ x\cos y + \partial_y g(y,z) \end{array}\right\} \Rightarrow g_y(y,z) = e^z$

$\Rightarrow$ integrating $\int e^z dy = ye^z + \text{const}$, we find $g(y,z) = ye^z + h(z)$.

• Finally, $ye^z = f_z(x,y,z) = ye^z + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = K$ - const

So: $\boxed{f(x,y,z) = x\sin y + ye^z + K}$ $\Rightarrow \int_C F\,d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = e + \sin 1$

Lecture #16

- ## Green's Theorem

! This result establishes the link b/w line integrals along closed simple curve C and double integral over the plane region D bounded by C.

<u>Convention</u>: Positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.

<u>Green Thm</u>: Let C be a positively oriented, piece-wise smooth, simple closed curve in the plane and let D be the region bounded by C. If P, Q have continuous partial derivatives on an open region that contains D, then:

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

<u>Rmks</u>: (1) Sometimes C is denoted by ∂D - boundary of D

(2) Also it is common to use $\oint_C$ to indicate the integral over a positively oriented closed curve C.

(3) When written in the form

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

this result is reminiscent of the Fundamental Thm Calculus

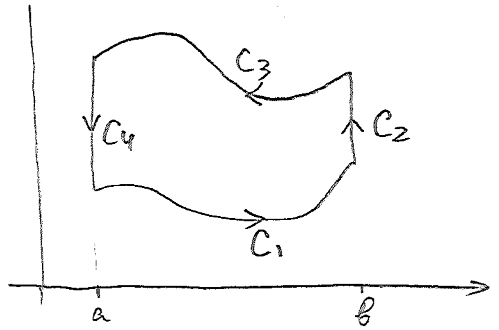
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(4) If $\langle P, Q\rangle$ is conservative, then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ and we recover that $\int_{\text{closed curve}} P\,dx + Q\,dy = 0$, so Green Theorem generalizes results from last time to the case of nonconservative vector fields

Lecture #16

- Sketch of the proof of Green Thm for special region D

Let us consider $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, $Q = 0$.



$$\iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\, dy\, dx \underset{\uparrow}{=} \int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right] dx$$

Fundamental Thm Calculus

$$\left.\begin{array}{l} \int_{C_1} P(x,y)\,dx = \int_a^b P(x, g_1(x))\,dx, \quad \int_{C_3} P(x,y)\,dx = -\int_{-C_3} P(x,y)\,dx = -\int_a^b P(x, g_2(x))\,dx \\ \int_{C_2} P(x,y)\,dx = 0 = \int_{C_4} P(x,y)\,dx \quad \text{as } dx = 0 \text{ on } C_2, C_4 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \int_C P(x,y)\,dx = \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx + \int_{C_3} P(x,y)\,dx + \int_{C_4} P(x,y)\,dx$$

$$= \int_a^b \left[P(x, g_1(x)) - P(x, g_2(x))\right] dx \overset{!}{=} -\iint_D \frac{\partial P}{\partial y}\, dA$$

$\blacksquare$

- Several variations of Green Theorem

Let us highlight a few variations of Green Theorem:

(1) If C is oriented clockwise, then we can use $\int_C P\,dx + Q\,dy = -\int_{-C} P\,dx + Q\,dy$ in combination with Green Theorem applied to $\int_{-C}$.

(2) If D is a finite union of simple regions then we can still apply Green Thm for each of them and sum up.

(3) If C is not closed, then you can apply Green Thm once you "complete C" to a closed curve by adding another C', which you choose so that $\int_{C'}$ is easy to compute

Lecture #16

Ex 4: Evaluate the line integral by two methods: (i) directly, (ii) using the Green's Theorem.

(a) $\oint_C (x^2+y)dx - y^3 dy$, C: the unit circle centered at the origin

(b) $\oint_C xy\,dx + x^2 dy$, C: rectangle w/ vertices (0,0), (3,0), (3,2), (0,2).

(a) Directly

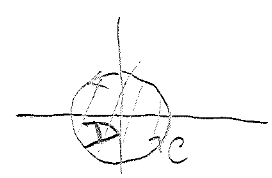
Parametrize $C: \vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

$$\oint_C (x^2+y)dx - y^3 dy = \int_0^{2\pi} [(\cos^2 t + \sin t)(-\sin t) - \sin^3 t \cdot \cos t]\,dt =$$

$$= -\int_0^{2\pi} \cos^2 t \sin t\,dt - \int_0^{2\pi} \sin^2 t\,dt - \int_0^{2\pi} \sin^3 t \cos t\,dt$$

But:
$$\left.\begin{aligned} &\int_0^{2\pi} -\cos^2 t \sin t\,dt \overset{u=\cos t}{=\!=} \int_1^1 u^2 du = 0 \\ &\int_0^{2\pi} \sin^3 t \cos t\,dt \overset{u=\sin t}{=\!=} \int_0^0 u^3 du = 0 \\ &\int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt = \pi \end{aligned}\right\} \Rightarrow \boxed{-\pi}$$

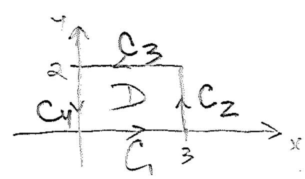
Green's Theorem



$P = x^2+y$, $Q = -y^3$ $\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1$

$$\Rightarrow \oint_C (x^2+y)dx - y^3 dy = \iint_D (-1)dA = -\iint_D 1\cdot dA = -\text{Area}(D) = \boxed{-\pi.}$$

(b) Green's Theorem



$P = xy$, $Q = x^2$ $\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x.$

So: $\oint_C xy\,dx + x^2 dy = \iint_D x\,dA = \int_0^3\int_0^2 x\,dy\,dx = \int_0^3 2x\,dx = x^2\Big|_0^3 = \boxed{9}$

Directly

Parametrize $C_1 = \{(t,0) \mid 0 \le t \le 3\}$, $C_2 = \{(3,t) \mid 0 \le t \le 2\}$, $C_3 = \{(t,2) \mid t \text{ from } 3 \text{ to } 0\}$,
$C_4 = \{(0,t) \mid t \text{ from } 2 \text{ to } 0\}$ and check $\int_{C_1} -''- = 0$, $\int_{C_2} -''- = 18$, $\int_{C_3} -''- = -9$, $\int_{C_4} -''- = 0$

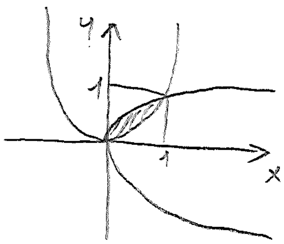
Lecture #16

Ex 5: Use Green's Theorem to evaluate the following integrals:

(a) $\oint_C (3y+e^{2\sqrt{x}})\,dx + (x+\sin y^3)\,dy$, C: boundary of the region bounded by $y=x^2$, $x=y^2$.

(b) $\oint_C y^4 dx + 2xy^3 dy$, C: ellipse $x^2+2y^2=2$

(a) $P = 3y + e^{2\sqrt{x}}$, $Q = x+\sin(y^3) \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2$

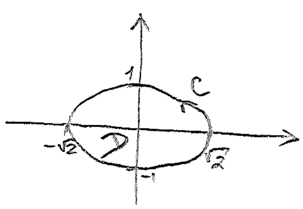


$$\oint_C (3y+e^{2\sqrt{x}})dx + (x+\sin(y^3))dy = \iint_D -2\,dA =$$

$$= -2\int_0^1 \int_{x^2}^{\sqrt{x}} 1\,dy\,dx = -2\int_0^1 (\sqrt{x}-x^2)dx = -2\left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= \boxed{-2/3}$$

(b) $P = y^4$, $Q = 2xy^3 \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2y^3$



$$\oint_C y^4 dx + 2xy^3 dy = \iint_D -2y^3\,dA =$$

$$= \int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} -2y^3\,dx\,dy = \int_{-1}^{1} -4y^3\sqrt{2-2y^2}\,dy$$

Use u-substitution $u = 2-2y^2 \Rightarrow \left.\begin{array}{l} du = -4y\,dy \\ y^2 = (1-u/2) \end{array}\right\} \Rightarrow$

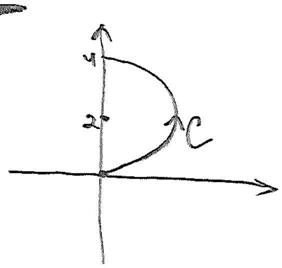
$$\Rightarrow \int_{-1}^{1} -4y^3\sqrt{2-2y^2}\,dy = \int_0^0 (1-\tfrac{u}{2})\sqrt{u}\,du = 0$$

So: $\oint_C y^4 dx + 2xy^3 dy = 0$.

Lecture #16

The next exercise must be discussed before Green's Theorem.

Ex6: Evaluate the line integral $\int_C e^{x^2}dx + \cos y\, dy$, where C is the right half circle of the radius 2 circle centered at $(0,2)$, starting from $(0,0)$ and terminating at $(0,4)$.



- Direct computation will fail as you will end up with an integral which is hard to compute.
- First, note that $F = \langle e^{x^2}, \cos y\rangle$ is conservative as $\frac{\partial}{\partial y}(e^{x^2}) = 0 = \frac{\partial}{\partial x}(\cos y)$.

But: Finding an explicit f-la for the potential f, s.t. $F = \nabla f$ is impossible as there is no explicit f-la for $\int e^{x^2}dx$!

Q: What should we do?

1st Proof

Since F is conservative, $\int_C F dr$ is path independent $\Rightarrow$

$\Rightarrow \int_C F dr = \int_{C'} F dr$, where C' is a line segment from $(0,0)$ to $(0,4)$.

Parametrize $C': \vec{r}(t) = \langle 0, t\rangle \ (0 \le t \le 4) \Rightarrow \int_{C'} e^{x^2}dx + \cos y\, dy = \int_0^4 \cos t\, dt = \boxed{\sin(4)}$

2nd Proof

Even though we don't have an explicit f-la $f(x,y)$, we know $f(x,y) = g(x) + \sin(y)$, where $g(x) = \int e^{x^2}dx$.

Hence: $\int_C F(x,y)d\vec{r} = \int_C \nabla f d\vec{r} = f(0,4) - f(0,0) = g(0) + \sin(4) - g(0) - \sin(0) = \boxed{\sin(4)}$