Lecture #17

10/31/2017

- Last time: Green's Theorem

Recall: $\boxed{\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$

where C – positively oriented closed simple curve, D – region bounded by C, and P, Q have continuous partial derivatives on the open region containing D.

* Sketch the proof in the simplest case - see p. 3 of Lecture #16.

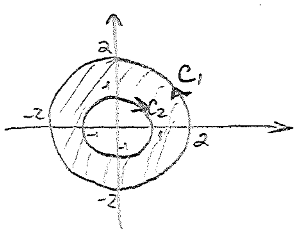
Ex1: Evaluate $\oint_C -y^3dx + x^3dy$, where C is the circle of radius 3, centered at the origin.

▸ $\oint_C -y^3dx + x^3dy = \iint_D (3x^2+3y^2)\,dA$, $D = \{(x,y) \mid x^2+y^2 \leq 9\}$ - disc enclosed by C.

$\iint_D 3(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^3 3\cdot r^2\cdot r\,dr\,d\theta = \frac{3}{4}r^4\Big|_0^3 \cdot 2\pi = \frac{243}{2}\pi$ ∎

Ex2: Evaluate $\oint_C (-y^4 + e^{2\cos x})dx + (2x^4 + e^{3\sin(y^2)})\,dy$, where C is the boundary of the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

▸



$C = C_1 \sqcup C_2$, where C_1 is oriented counter clockwise
C_2 is oriented clockwise

(recall that orientation is chosen so that as we move along the curve, our region is on the left).

$\oint_C (-y^4+e^{2\cos x})dx + (2x^4+e^{3\sin(y^2)})\,dy = \iint_D (8x^3+4y^3)\,dA$, where D in polar coordinates is $D = \{(r,\theta) \mid 0 \leq \theta \leq 2\pi,\ 1 \leq r \leq 2\}$

So: $\iint_D (8x^3+4y^3)\,dA = \int_0^{2\pi}\int_1^2 (8r^3\cos^3\theta + 4r^3\sin^3\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\left(\int_1^2 (8r^4\cos^3\theta + 4r^4\sin^3\theta)\,dr\right)d\theta$

$= \int_0^{2\pi}\left(\frac{8}{5}\cdot 31\cos^3\theta + \frac{4}{5}\cdot 31\cdot\sin^3\theta\right)d\theta = \frac{248}{5}\int_0^{2\pi}\cos^3\theta\,d\theta + \frac{124}{5}\int_0^{2\pi}\sin^3\theta\,d\theta$

$\int_0^{2\pi}\cos^3\theta\,d\theta = \int_0^{2\pi}\cos^2\theta\cdot\cos\theta\,d\theta = \int_0^{2\pi}(1-\sin^2\theta)\cos\theta\,d\theta = \sin\theta\Big|_0^{2\pi} - \frac{\sin^3\theta}{3}\Big|_0^{2\pi} = 0$

$\int_0^{2\pi}\sin^3\theta\,d\theta = \int_0^{2\pi}\sin^2\theta\cdot\sin\theta\,d\theta = \int_0^{2\pi}(1-\cos^2\theta)\sin\theta\,d\theta = -\cos\theta\Big|_0^{2\pi} + \frac{\cos^3\theta}{3}\Big|_0^{2\pi} = 0.$

So: $\oint_C (-y^4+e^{2\cos x})dx + (2x^4+e^{3\sin(y^2)})\,dy = 0$ ∎

Rmk: We could also argue that $\oint_C = \oint_{C_1} - \oint_{-C_2}$, where both $C_1, -C_2$ bound disks.

Lecture #17

Note that Green Theorem can be also used in the other way: reducing double integral to line integrals.

Key Example: Area $(D) = \iint_D 1\,dA = \oint_{\partial D} P(x,y)\,dx + Q(x,y)\,dy$ for any P, Q s.t. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

$$\Downarrow$$

$$\boxed{\text{Area } (D) = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

Ex3: Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

► Let C be the boundary of this region, i.e. $C = \{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$.
It can be parametrized by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.

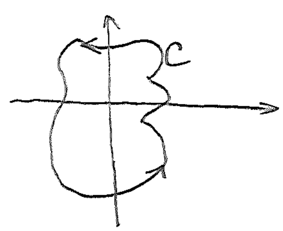
So: $\text{Area}(D) = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt =$

$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2 t + \sin^2 t)\,dt = \pi ab$. ■

! Ex4: Let C be any positively oriented simple closed path that encloses the origin. Evaluate

$$\oint_C F\,dr, \quad F = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

►



The key differences from the previous cases:

- C is arbitrary (not a fixed one)
- the components of F are not continuous on the whole region enclosed by C.

Note however that if $P = \frac{-y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}$, then on $\mathbb{R}^2 \setminus \{(0,0)\}$, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} - \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0.$$

! However, the Fundamental Theorem of Line Integrals can NOT be applied as P, Q do not have continuous partials on the whole region D enclosed by C.

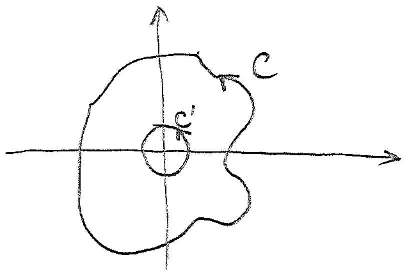
Q: What should we do to compute $\oint_C$?

A: Reduce to the case of C being a simple curve, when $\oint_C$ is computable directly.

Lecture #17

(Continuation).

Let C' be a counterclockwise oriented small circle centered at the origin and lying inside the region D (bounded by C).



<u>Then</u>: $\oint_{C\cup(-C')} Pdx+Qdy = \iint_{D'}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$, where D' is the region bounded by C' and C.

As we checked above, since $(0,0)\notin D'$, we have $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$ on D'

<u>Therefore</u>: $\oint_{C\cup(-C')} Pdx+Qdy = 0 \Rightarrow \oint_C Pdx+Qdy = \oint_{C'} Pdx+Qdy$.

As C' is a circle, the latter integral can be computed directly.

Parametrize C': $(r\cos t, r\sin t)$, $0\le t\le 2\pi$, r - radius of C'.

<u>Then</u>: $\oint_{C'} F\,dr = \int_0^{2\pi}\left[\frac{-r\sin t}{r^2}(-r\sin t) + \frac{r\cos t}{r^2}(r\cos t)\right]dt = \int_0^{2\pi} dt = 2\pi$.

<u>Upshot</u>: $\oint_C F\,dr = 2\pi$ (is independent of C) ∎

<u>Warning</u>: This example illustrates that when applying the fundamental theorem of line integrals it is not sufficient to check $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$, <u>But</u> you also need to make sure P, Q have partials continuous on D.

! Now we can finish the proof of thm from last tuesday:

<u>Thm</u>: Let $F = P\vec{i}+Q\vec{j}$ be a vector field on an open simply-connected region D, s.t. P, Q have continuous 1st order partial derivatives and $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ on D. Then F is conservative.

► It suffices to prove that $\int_C F\,dr$ is path-independent or alternatively, $\int_C F\,dr = 0$ for any closed C. If C is simple, then $\oint_C F\,dr = \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \iint_D 0\,dA = 0$

If C is not simple, i.e. has several self-intersections, then we can still break it up into several simple curves and sum corresponding integrals which are ZERO ∎

Lecture #17

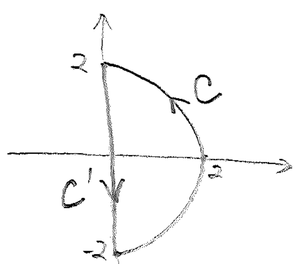
Ex 5: Evaluate $\int_C (y+e^{\sin(x^3)})dx + x^2 dy$, where C is the half circle centered at the origin of radius 2, which is traversed clockwise from the point $(0,-2)$ to $(0,2)$.

- First, let me note that if there was no $e^{\sin(x^3)}$ you could compute this line integral directly by parametrizing $C: (2\cos\theta, 2\sin\theta)$, $-\pi/2 \le \theta \le \pi/2$.

  But, in the presence of $e^{\sin(x^3)}$ you would need to integrate $\int_{-\pi/2}^{\pi/2} e^{\sin((2\cos\theta)^3)} \cdot (-2\sin\theta)d\theta$, which is impossible on the nose.

- Second, we check if the v.field $F = \langle y+e^{\sin(x^3)}, x^2\rangle$ is conservative, but it is not.

- Hence, the only remaining tool is the Green Theorem. But to apply the latter, we need to complete C to a closed curve. The easiest way is by connecting $(0,2)$ to $(0,-2)$ by a line segment.



Then: $\oint_{C\cup C'} (y+e^{\sin(x^3)})dx + x^2 dy \overset{\text{Green}}{=} \iint_D (2x-1)dA =$

$$= \int_{-\pi/2}^{\pi/2}\int_0^2 (2r\cos\theta - 1)r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(2\cos\theta\cdot\frac{r^3}{3}\Big|_0^2 - \frac{r^2}{2}\Big|_0^2\right)d\theta$$

$$= \frac{16}{3}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta - 2\pi = \boxed{\frac{32}{3} - 2\pi}$$

So far, we computed $\oint_{C\cup C'} (y+e^{\sin(x^3)})dx + x^2 dy$ instead of $\int_C$ –॥–.

However, we have $\oint_{C\cup C'}$ –॥– $= \int_C$ –॥– $+ \int_{C'}$ –॥–.

So: $\int_C (y+e^{\sin(x^3)})dx + x^2 dy = \frac{32}{3} - 2\pi - \int_{C'} (y+e^{\sin(x^3)})dx + x^2 dy$.

Parametrize C': $(0,t)$, t ranging from 2 to -2. Then $dx=0$, $dy=dt$, but $x\equiv 0$ on C', so you end up with $\int_{C'}$ –॥– $= \int_2^{-2} 0\,dt = 0$.

Thus: $\int_C (y+e^{\sin(x^3)})dx + x^2 dy = \frac{32}{3} - 2\pi$ ∎

Lecture #17

Rmk: The idea used in Ex5 is very similar to the last Exercise from Thursday, when we couldn't find the potential of the conservative vector field explicitly, but we got rid of this by changing the path.

Idea: If you start from a non-closed curve C b/w points A to B and you don't know how to evaluate $\int_C F dr$, you can try to choose a simpler path C' b/w A & B for which $\int_{C'} F dr$ can be directly computed and (by parametrizing C')

then also compute the difference $\int_C F dr - \int_{C'} F dr = \int_{C \cup (-C')} F dr$.

The latter is an integral over a closed curve $C \cup (-C')$.

Therefore, if your vector field is conservative, you get ZERO, but if not conservative, you can evaluate it via Green Thm.