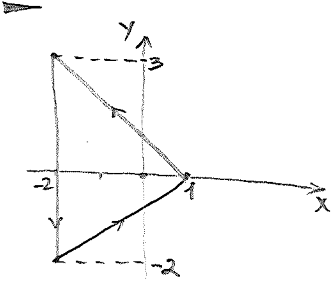
Lecture #18 11/02/2017

- Last time
  - We saw that before applying the FTLI to the field with $Q_x = P_y$, one needs to check whether these derivatives are continuous everywhere on the region. (not only the curve)
  - If your region contains several "bad" points, where first order partial derivatives are not continuous, you have to "eliminate" them by computing line integrals along small circles around them by hand
  - If your curve is not closed, to apply Green's Theorem you must CLOSE the curve first.

Ex1: Evaluate the integral $\oint_C F\,dr$, where

$$F = \left(-\frac{5y}{x^2+y^2} + y^2\right)\vec{i} + \left(\frac{5x}{x^2+y^2} + xy\right)\vec{j}$$

C: boundary of the triangle with vertices at $(1,0)$, $(-2,3)$, $(-2,-2)$.



Note that the components $P(x,y) = -\frac{5y}{x^2+y^2} + y^2$, $Q(x,y) = \frac{5x}{x^2+y^2} + xy$ have continuous first order partial derivatives everywhere except the origin $(0,0)$, while at the origin even P, Q themselves are not well-defined.

As a result, we cannot apply Green's Thm on the nose.

However: We can split F as $F = F_1 + F_2$, where $F_1 = -\frac{5y}{x^2+y^2}\vec{i} + \frac{5x}{x^2+y^2}\vec{j}$, $F_2 = y^2\vec{i} + xy\vec{j}$ and use $\oint_C F\,dr = \oint_C F_1\,dr + \oint_C F_2\,dr$.

Note that C is a simple closed curve enclosing the origin, hence, using exercise from last time $\oint_C F_1\,dr = 5\cdot 2\pi = \boxed{10\pi}$.

Meanwhile, $\oint_C F_2\,dr$ can be computed either directly or via Green.

Green's Thm: $\oint_C F_2\,dr = \iint_D -y\,dA \overset{*}{=} \int_{-2}^{1}\int_{\frac{2}{3}x-\frac{2}{3}}^{1-x} -y\,dy\,dx = \int_{-2}^{1} -\frac{y^2}{2}\Big|_{\frac{2}{3}x-\frac{2}{3}}^{1-x} dx =$

$= \int_{-2}^{1} -\frac{1}{2}\left(1-2x+x^2-\frac{4}{9}x^2+\frac{8}{9}x-\frac{4}{9}\right)dx = -\frac{1}{2}\int_{-2}^{1}\left(\frac{5}{9}-\frac{10}{9}x+\frac{5}{9}x^2\right)dx = -\frac{1}{2}\left(\frac{5}{9}\cdot 3 - \frac{10}{18}\cdot(-3) + \frac{5}{27}\cdot 9\right)$

$= -\frac{1}{2}\left(\frac{5}{3}+\frac{5}{3}+\frac{5}{3}\right) = \boxed{-\frac{5}{2}}$

So: $\boxed{\oint_C F\,dr = 10\pi - \frac{5}{2}}$

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• Today: "Curl" and "Divergence"

Def 1: Given a vector field $F = P\vec{i} + Q\vec{j} + R\vec{k}$ on $\mathbb{R}^3$, s.t. partial derivatives of P, Q, R exist, then curl of F is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{\operatorname{curl}(F) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}}$$

To simplify this formula, let us use the following "cheating notation":

$$\boxed{\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}}$$

(i.e. it is a vector field, whose components are differential operators)

for any f-n f, ∇f is just the gradient vector field of f.

Let us now compute the cross-product

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \underbrace{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}}_{\operatorname{curl}(F).}$$

Upshot: $\boxed{\operatorname{curl} F = \nabla \times F}$

Def 2: Given a vector field $F = P\vec{i} + Q\vec{j} + R\vec{k}$ on $\mathbb{R}^3$, s.t. $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ exist, then divergence of F is the function of 3 variables given by

$$\boxed{\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Analogously to above, note that $\nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \operatorname{div}(F)$.

So: $\boxed{\operatorname{div} F = \nabla \cdot F}$

Ex 2: Find the curl and divergence of the following vector field

$$F = xye^z\,\vec{i} + \sin(yz)\,\vec{j} + xze^y\,\vec{k}$$

► $\operatorname{div} F = \nabla \cdot F = ye^z + z\cdot\cos(yz) + xe^y$

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xye^z & \sin(yz) & xze^y \end{vmatrix} = (xze^y - y\cos(yz))\vec{i} - (ze^y - xye^z)\vec{j} + (0 - xe^z)\vec{k}$$

■

Thm: (1) curl $(\nabla f)=0$ for any function f that has continuous second order partial derivatives.

(2) If F is a vector field defined on all $\mathbb{R}^3$, whose components have continuous partial derivatives and curl $F=0$, then F is a conservative vector field, i.e. $F=\nabla f$ for some f.

! Assign (1) as an exercise.

(1) $\text{curl}(\nabla f)=\nabla\times(\nabla f)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{vmatrix} = \underbrace{\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)}_{=0}\vec{i}-\underbrace{\left(\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial^2 f}{\partial z\partial x}\right)}_{=0}\vec{j}+\underbrace{\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)}_{=0}\vec{k}$

(2) Will be proved later in the course. ∎

Ex3: Determine whether or not the vector field is conservative. If it is conservative, find potential f s.t. $F=\nabla f$.

(a) $F=e^y\cos x\cdot\vec{i}+xye^z\vec{j}+z\sin(y)\vec{k}$

(b) $F=(1+e^y+ze^x)\vec{i}+xe^y\vec{j}+e^x\vec{k}$

Rmk: Note that this is basically the same exercise as we had some time ago.

(a) If F was conservative, we would have curl$(F)=0$ by Theorem above. However, when we start computing curl$(F)=\nabla\times F$, we see that already a coefficient of $\vec{i}$ is $z\cdot\cos y-xye^z\neq 0$. Contradiction!

So: F is $\boxed{\text{not conservative}}$

(b) $\text{curl}(F)=\nabla\times F=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1+e^y+ze^x & xe^y & e^x\end{vmatrix}=(0-0)\vec{i}-(e^x-e^x)\vec{j}+(e^y-e^y)\vec{k}=0.$

Hence, by part (2) of Thm above, F is $\boxed{\text{conservative}}$, i.e. $F=\nabla f$ for some f and we find this f in the same way as before.

* $\int e^x dz=ze^x+\text{const}\Rightarrow f(x,y,z)=ze^x+g(x,y)$
* $xe^y=g_y(x,y)\Rightarrow g(x,y)=\int xe^y dy=xe^y+h(x)$

$\Big\}\Rightarrow f(x,y,z)=ze^x+xe^y+h(x)$

* $1+e^y+ze^x=ze^x+e^y+h'(x)\Rightarrow h(x)=x+K.$

So: $\boxed{f(x,y,z)=ze^x+xe^y+x+K}$ ∎

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Ex 4: Compute div F, where $F = \langle xze^y - y\cos(yz),\ xye^z - ze^y,\ -xe^z \rangle$
the curl of the vector field from Ex2.

$$\operatorname{div} F = ze^y + (xe^z - ze^y) - xe^z = 0$$ ∎

This ZERO is not accidental as we have:

Thm: If $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second-order partial derivatives, then

$$\boxed{\operatorname{div} \operatorname{curl} F = 0}$$

$$\operatorname{div} \operatorname{curl} F = \operatorname{div}\left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right)$$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0$$ ∎

Ex 5: Determine whether or not the vector field F is a curl of some other vector field G. If it is, find any G such that F = curl G.

(a) $F = \langle x\sin z,\ y^2 + xz,\ x + \cos z \rangle$

(b) $F = \langle -y, z, y \rangle$

(a) If $F = \operatorname{curl} G$, then by above theorem we must have $\operatorname{div} F = 0$.
However, $\operatorname{div} F = \sin z + 2y - \sin z = 2y \neq 0$. So: the answer is $\boxed{\text{NO}}$

(b) Similarly, compute first div F: $\operatorname{div} F = 0 + 0 + 0 = 0$.
Let us now try to find $G = \langle P, Q, R \rangle$ so that $F = \operatorname{curl}(G)$.

* First, we start from $R_y - Q_z = y$. Choose R any way, the simplest choice is $R = 0 \Rightarrow -Q_z = y \Rightarrow Q = yz + g(x,y)$.

* Analogously $P_z - R_x = z \overset{R=0}{\Rightarrow} P_z = z \Rightarrow P = \frac{z^2}{2} + h(x,y)$

* Finally $Q_x - P_y = y \Rightarrow g_x(x,y) - h_y(x,y) = y$.
For example, take $h(x,y) = 0$, $g(x,y) = xy$

Thus: $\boxed{F = \operatorname{curl}(\langle z^2/2,\ -yz + xy,\ 0 \rangle)}$! There are too many choices! ∎

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Ex6: Prove the identity, assuming the appropriate partial derivatives exist and are continuous

$$\mathrm{div}(F\times G) = G\cdot \mathrm{curl}(F) - F\cdot \mathrm{curl}(G)$$

► Let $F = \langle P_1, Q_1, R_1\rangle$, $G = \langle P_2, Q_2, R_2\rangle$.

Then $F\times G = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = \langle Q_1R_2 - Q_2R_1,\ P_2R_1 - P_1R_2,\ P_1Q_2 - P_2Q_1\rangle$

$$\Rightarrow \mathrm{div}(F\times G) = \frac{\partial(Q_1R_2 - Q_2R_1)}{\partial x} + \frac{\partial(P_2R_1 - P_1R_2)}{\partial y} + \frac{\partial(P_1Q_2 - P_2Q_1)}{\partial z}$$

$$= \frac{\partial Q_1}{\partial x}\cdot R_2 - \frac{\partial R_1}{\partial x}\cdot Q_2 + \frac{\partial R_2}{\partial x}Q_1 - \frac{\partial Q_2}{\partial x}R_1$$
$$+ \frac{\partial R_1}{\partial y}\cdot P_2 - \frac{\partial P_1}{\partial y}\cdot R_2 + \frac{\partial P_2}{\partial y}R_1 - \frac{\partial R_2}{\partial y}\cdot P_1$$
$$+ \frac{\partial P_1}{\partial z}\cdot Q_2 - \frac{\partial Q_1}{\partial z}\cdot P_2 + \frac{\partial Q_2}{\partial z}P_1 - \frac{\partial P_2}{\partial z}\cdot Q_1$$

$$= P_2\cdot\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)$$
$$- P_1\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) - Q_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) - R_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)$$
$$= G\cdot \mathrm{curl}(F) - F\cdot \mathrm{curl}(G)$$

■

Let us conclude by obtaining vector forms of Green's Theorem.

① Let us start from the region D, its boundary $C = \partial D$ and two functions P, Q satisfying assumptions of Green's Theorem. Place the region $D \subset \mathbb{R}^2 \underset{xy\text{-plane}}{\subseteq} \mathbb{R}^3$.

Let $F = P\vec{i} + Q\vec{j}$ be the vector field on $\mathbb{R}^2$, which can be also viewed as a vector field on $\mathbb{R}^3$ with ZERO third component.

Then $\mathrm{curl}\, F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (Q_x - P_y)\vec{k}$ (as $Q_z = 0 = P_z$)

<u>So</u>: the Green's Theorem can be rewritten as

$$\boxed{\oint_C F\, d\vec{r} = \iint_D (\mathrm{curl}\, F)\cdot \vec{k}\, dA}$$

Note that in the LHS we have kind of "tangential" component of F, while in RHS we have "normal" component of curl F.

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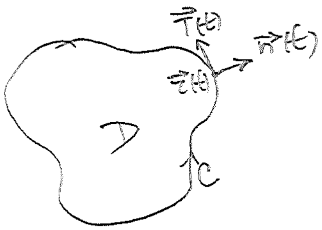
(2) While the LHS of previous equality, $\oint_C F dr$, took care of the tangential component of the vector field, let us now compute the integral of its normal component.

Let us parametrize the curve $C: \vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$.
Then the unit tangent vector is $\vec{T}(t) = \frac{x'(t)}{|\vec{r}'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}'(t)|}\vec{j}$, hence, the outward unit normal vector to C is $\vec{n}(t) = \frac{y'(t)}{|\vec{r}'(t)|}\vec{i} - \frac{x'(t)}{|\vec{r}'(t)|}\vec{j}$.



Let us now evaluate

$$\oint_C F \cdot \vec{n}\, ds = \int_a^b (F \cdot \vec{n})(t) \cdot |\vec{r}'(t)|\, dt =$$

$$= \int_a^b \frac{P(x(t), y(t)) \cdot y'(t) - Q(x(t), y(t)) \cdot x'(t)}{\cancel{|\vec{r}'(t)|}} \cdot \cancel{|\vec{r}'(t)|}\, dt$$

$$= \int_a^b P dy - Q dx \overset{\text{Green}}{=\!=} \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_D \operatorname{div} F(x,y)\, dA.$$

<u>So</u>: $\boxed{\oint_C F \cdot \vec{n}\, ds = \iint_D \operatorname{div} F(x,y)\, dA}$

The obtained two formulas

$$\boxed{\oint_C F dr = \iint_D (\operatorname{curl} F) \cdot \vec{k}\, dA \quad \text{and} \quad \oint_C F \cdot \vec{n}\, ds = \iint_D \operatorname{div} F(x,y)\, dA}$$

will be generalized later on in this class.

<u>Rmks</u>: (1) The former of this will be generalized to the case when D is not a region in $\mathbb{R}^2 \subset \mathbb{R}^3$, but rather a piece-wise oriented surface ~~in~~ $\mathbb{R}^3$, while $\vec{k}$ should be replaced by a normal vector at that point.

(2) The latter would be generalized to the higher dimension. The reason is that $\vec{n}$ is not defined for a curve in $\mathbb{R}^3$. However, $\vec{n}$ is well-defined for a surface in $\mathbb{R}^3$ and so the LHS should be replaced by a double integral while the RHS will be replaced by a triple integral