Lecture #19 11/07/2017

- Follow pp. 5-6 from Lecture Notes of Lecture #18, where we proved:

$$\boxed{\oint_C F\,dr = \iint_D (\text{curl } F)\cdot \vec{k}\, dA} \quad \text{and} \quad \boxed{\oint_C F\cdot\vec{n}\, ds = \iint_D \text{div } F(x,y)\, dA}$$

- Today: Parametric surfaces

Analogously to curves, we can describe a surface by a vector function $\vec{r}(u,v)$ of two parameters u, v:

$$\boxed{\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}}$$ – vector-valued function defined on $D \subseteq \mathbb{R}^2_{uv}$

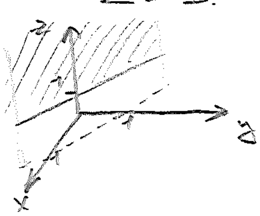
Def: The set of all points $(x,y,z) \in \mathbb{R}^3$ such that $x = x(u,v), y = y(u,v), z = z(u,v)$, for some $(u,v) \in D$, is called a parametric surface.

Ex1: Identify and sketch the following surfaces with vector equation:

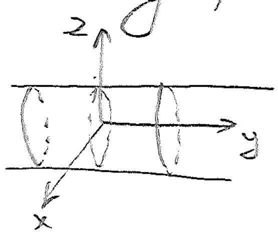
(a) $\vec{r}(u,v) = (1-2u)\vec{i} + 2u\vec{j} + e^{v^2}\vec{k}$

(b) $\vec{r}(u,v) = 5\sin u\cdot\vec{i} - 10v\vec{j} + 5\cos u\cdot\vec{k}$

(a) The corresponding surface is the part of the plane $x+y=1$, where $z \geq 1$.



(b) The corresponding surface is the cylinder whose base is a circle of radius 5 in the xz-plane (with center at the origin) and whose axis is parallel to y-axis:



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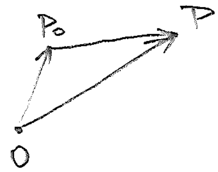
Def: If we keep u constant by putting $u=u_0$, then $\vec{r}(u_0,v)$ defines a curve C_1 on S. Likewise, if we keep v constant by putting $v=v_0$, then $\vec{r}(u,v_0)$ defines a curve C_2 on S. These curves are called grid curves

Ex2: Find a parametric representation of the plane passing through the point $P_0(x_0,y_0,z_0)$ and which contains two non-parallel vectors $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$.



$$\vec{OP}=\underbrace{\vec{OP_0}}_{\langle x_0,y_0,z_0\rangle}+\underbrace{\vec{P_0P}}_{u\vec{a}+v\vec{b}\ (u,v\in\mathbb{R})}$$

Hence:
$$\begin{cases} x=x_0+ua_1+vb_1 \\ y=y_0+ua_2+vb_2 \\ z=z_0+ua_3+vb_3. \end{cases} \quad u,v\in\mathbb{R}$$

Ex3: Find a parametric equation of the cylinder $x^2+y^2=9$, $-2\le z\le 5$.

$x=3\cos\theta$, $y=3\sin\theta$, $z=z$, where $0\le\theta\le 2\pi$, $-2\le z\le 5$

Ex4: Find a parametric equation of a graph of a function $f(x,y)$ on $\mathbb{R}^2$.

$x=x$, $y=y$, $z=f(x,y)$ where $x,y\in\mathbb{R}$.

Ex5: Find a parametric equation of the top half (i.e. $z\ge 0$) of the cone $z^2=16(x^2+y^2)$.

1st way: $x=x$, $y=y$, $z=4\sqrt{x^2+y^2}$, $x,y\in\mathbb{R}$

2nd way: $x=r\cos\theta$, $y=r\sin\theta$, $z=4r$, $r\ge 0$, $0\le\theta\le 2\pi$

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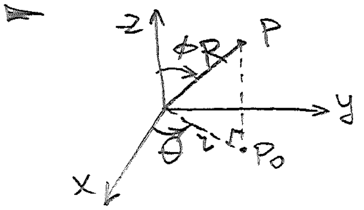
Ex6: Find a parametric equation of a sphere $x^2+y^2+z^2=R^2$.



$$\left.\begin{aligned} x&=r\cos\theta \\ y&=r\sin\theta \\ r&=R\sin\phi \end{aligned}\right\} \Rightarrow \begin{aligned} x&=R\sin\phi\cos\theta \\ y&=R\sin\phi\sin\theta \end{aligned}$$

Also $z=R\cos\phi$

So: $\begin{cases} x=R\sin\phi\cos\theta \\ y=R\sin\phi\sin\theta \\ z=R\cos\phi \end{cases}$, where $\begin{aligned} 0\le\theta\le 2\pi \\ 0\le\phi\le\pi \end{aligned}$

Note that it is not 2π!

Ex7 (Surfaces of revolution): Let S be obtained by rotating the curve $y=f(x)$, $a\le x\le b$ (for simplicity assume $f(x)\ge 0$) about the x-axis. Find a parametric equation of S.

$x=x$, $y=f(x)\cos\theta$, $z=f(x)\sin\theta$, where $a\le x\le b$, $0\le\theta\le 2\pi$

Tangent Planes

Want: Find the tangent plane to the parametric surface S traced out by $\vec{r}(u,v)=\langle x(u,v), y(u,v), z(u,v)\rangle$ at the point $P_0=\vec{r}(u_0,v_0)$.

Idea: We know that tangent plane contains all the tangent vectors to various curves on S passing through P_0.

Of particular interest are the grid curves C_1 & C_2 from p.2.

The tangent vector to C_1 at P_0 is $\vec{r}_v=\frac{\partial x}{\partial v}(u_0,v_0)\vec{i}+\frac{\partial y}{\partial v}(u_0,v_0)\vec{j}+\frac{\partial z}{\partial v}(u_0,v_0)\vec{k}$

The tangent vector to C_2 at P_0 is $\vec{r}_u=\frac{\partial x}{\partial u}(u_0,v_0)\vec{i}+\frac{\partial y}{\partial u}(u_0,v_0)\vec{j}+\frac{\partial z}{\partial u}(u_0,v_0)\vec{k}$

Def: S is *smooth* if $\vec{r}_u\times\vec{r}_v\ne 0$.

Upshot: If S is smooth, then the tangent plane at P_0 has $\vec{r}_u\times\vec{r}_v$ as a normal vector.

! Hand out the worksheet with matching game.