• Tangent Planes

Goal: Find the tangent plane to the parametric surface S traced out by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ at the point $P_0 = \vec{r}(u_0,v_0)$.

As we know, the tangent plane contains all tangent vectors (at P_0) to various curves on S passing through P_0. Of particular interest are the so-called grid curves.

Fix $u=u_0 \rightsquigarrow C_1: \vec{r}(u_0,v)$ – a curve on S passing through P_0.
The tangent vector to C_1 at P_0 is obtained by differentiating by v the position vector $\vec{r}(u_0,v)$: $\boxed{\vec{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\vec{k}}$

Fix $v=v_0 \rightsquigarrow C_2: \vec{r}(u,v_0)$ – a curve on S passing through P_0
The tangent vector to C_2 at P_0 is obtained by differentiating by u the position vector $\vec{r}(u,v_0)$: $\boxed{\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)\vec{k}}$

Def: S – smooth if $\vec{r}_u \times \vec{r}_v \neq 0$

If S is smooth, then the tangent plane to S at P_0 is determined by:
(1) it passes through P_0
(2) it has $\vec{r}_u \times \vec{r}_v$ as a normal vector.

Ex1: Describe the surface S parametrized by $\vec{r}(u,v) = u^2\vec{i} + 2u\sin(v)\vec{j} + u\cos v\vec{k}$.
Find the tangent plane to S at the point P_0 given by $\vec{r}(1,0)$ $\left(\begin{smallmatrix} u_0=1 \\ v_0=0\end{smallmatrix}\right)$

(a) Note that $(2u\sin v)^2 + 4\cdot(u\cos v)^2 = 4u^2 \Rightarrow 4x = y^2+4z^2$ – elliptic paraboloid.
It is easy to see that any (x,y,z) satisfying $4x=y^2+4z^2$ is on S.

(b) $\vec{r}_u = 2u_0\vec{i} + 2\sin(v_0)\vec{j} + \cos(v_0)\vec{k} = 2\vec{i} + \vec{k}$

$\vec{r}_v = 2u_0\cos(v_0)\vec{j} - u_0\sin(v_0)\vec{k} = 2\vec{j}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2\vec{i} + 4\vec{k}$$

Hence, the eq-n of the tangent plane is $-2(x-x_0)+4(z-z_0)=0$, where $\langle x_0,y_0,z_0\rangle = \vec{r}(u_0,v_0) = \langle 1,0,1\rangle \Rightarrow$ get $-2(x-1)+4(z-1)=0 \Rightarrow -2x+2+4z-4=0$.

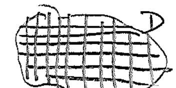
$\Rightarrow \boxed{x-2z=-1}$

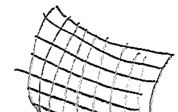
Lecture #20

• Surface Integrals

Goal: Integrate a function $f(x,y,z)$ over the surface S given by the parametric equation

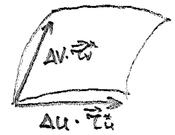
$$\boxed{\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}\ , (u,v)\in D}$$

Idea: If we split D into small rectangles R_{ij} 

with dimensions $\Delta u, \Delta v$, then the surface S is divided into corresponding patches S_{ij} 

And we want to approximate $\sum_{i,j} \text{Area}(S_{ij}) \cdot f(P^*_{ij})$ (P^*_{ij}: point on S_{ij}.)

Key observation: The ~~patch~~ patch S_{ij} can be approximated by the vectors $\Delta u \cdot \vec{r}_u^*$ and $\Delta v \cdot \vec{r}_v^*$. Recall that the area of a parallelogram can be computed via cross-product, so that



$$\text{Area}(S_{ij}) \approx \Delta u \cdot \Delta v \cdot |\vec{r}_u^* \times \vec{r}_v^*| \quad \left(\begin{array}{c} ^* \text{ denotes} \\ \text{evaluated at} \\ \text{left-bottom corner}\end{array}\right)$$

This motivates the following definition

Def 1: If a smooth parametric surface is given by the equation

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}\ , (u,v)\in D,$$

and S is <u>covered just once</u> as (u,v) ranges through D, then the *surface area of S* is

$$\boxed{A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA}$$

More generally, we have

Def 2: Under the same conditions and notations as in Def 1, the *surface integral* of f over the surface S is

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v|\, dA}$$

Lecture #20

Rmks: (1) $\iint_S 1\, dS = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA = \text{Area}(S)$

(2) Note the analogy with the line integral
$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\cdot|\vec{r}\,'(t)|\,dt.$$

Ex2: (a) Find the surface area of a sphere S of radius R.

(b) Find the surface integral $\iint_S x^2\,dS$, where S is the same as in (a).

(a) Recall the parametrization of a sphere in spherical coordinates from last lecture:

$x = R\sin\phi\cos\theta,\quad y = R\sin\phi\sin\theta,\quad z = R\cos\phi,\qquad D = \{(\phi,\theta) \mid 0\le\phi\le\pi,\ 0\le\theta\le 2\pi\}$

$\vec{r}_\phi = \langle R\cos\phi\cos\theta,\ R\cos\phi\sin\theta,\ -R\sin\phi\rangle$

$\vec{r}_\theta = \langle -R\sin\phi\sin\theta,\ R\sin\phi\cos\theta,\ 0\ \rangle$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix} = \vec{i}\cdot R^2\sin^2\phi\cos\theta + \vec{j}\cdot R^2\sin^2\phi\sin\theta + \vec{k}\cdot R^2\sin\phi\cos\phi.$$

$$\Rightarrow |\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{R^4\sin^4\phi(\cos^2\theta+\sin^2\theta) + R^4\sin^2\phi\cos^2\phi} = R^2|\sin\phi| = R^2\sin\phi \quad (\text{as } 0\le\phi\le\pi)$$

So: $\text{Area}(S) = \iint_D R^2\sin\phi\,dA = \int_0^{2\pi}\int_0^{\pi} R^2\sin\phi\,d\phi\,d\theta = \boxed{4\pi R^2}$

(b) As in part (a), we get:

$$\iint_S x^2\,dS = \iint_D R^2\sin^2\phi\cos^2\theta\, R^2\sin\phi\,dA = R^4\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\cos^2\theta\,d\phi\,d\theta$$
$$= R^4\cdot\int_0^{2\pi}\cos^2\theta\,d\theta\cdot\int_0^{\pi}\sin^3\phi\,d\phi = R^4\cdot\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta\cdot\int_0^{\pi}(1-\cos^2\phi)\cdot d(\cos\phi)$$
$$= R^4\cdot\pi\cdot\int_{-1}^{1}(1-u^2)\,du = R^4\cdot\pi\cdot\left(u-\frac{u^3}{3}\right)\Big|_{u=-1}^{u=1} = \boxed{\frac{4}{3}\pi R^4}$$

Remark: One way to view the surface integral is as follows. Consider the thin sheet of aluminium foil that has shape of a surface S and the density at point (x,y,z) is $\rho(x,y,z)$, then the total mass of the sheet is $m = \iint_S \rho(x,y,z)\,dS$, while the coordinates of the center of mass are:

$$\left(\frac{1}{m}\iint_S x\rho(x,y,z)\,dS,\ \frac{1}{m}\iint_S y\rho(x,y,z)\,dS,\ \frac{1}{m}\iint_S z\rho(x,y,z)\,dS\right)$$

Lecture #20

- **Special case: graphs of functions**

One of the most common examples of surfaces are graphs of functions in two variables. In other words, let S be the graph of $f(x,y)$, $(x,y)\in D$. The canonical parametrization of S is:

$$\boxed{\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}, \quad (x,y)\in D}$$

Then: $\left.\begin{array}{l}\vec{r}_x = \vec{i} + f_x\cdot\vec{k} \\ \vec{r}_y = \vec{j} + f_y\cdot\vec{k}\end{array}\right\} \Rightarrow \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \boxed{-f_x\cdot\vec{i} - f_y\cdot\vec{j} + \vec{k}}$

Therefore: $\boxed{\text{Area}(S) = \iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dA}$

and more generally

$$\boxed{\iint_S g(x,y,z)\,dS = \iint_D g(x,y,f(x,y))\cdot\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dA}$$

Rmk: Similar formulas apply when S is realized as a graph of functions in x,z or y,z, i.e. when $y=f(x,z)$ or $x=f(y,z)$.

Ex3: Find the area of the part S of the paraboloid $y=x^2+z^2$ that lies within the cylinder $x^2+z^2=16$.

► $\text{Area}(S) = \iint_{D:\,x^2+z^2\le 16} \sqrt{1+(2x)^2+(2z)^2}\, dA = \iint_D \sqrt{1+4(x^2+z^2)}\, dA.$

Switching to polar coordinates (r,θ) instead of (x,z), we get:

$$\text{Area}(S) = \int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\, r\, dr d\theta \underset{u=1+4r^2}{=} \int_0^{2\pi}\int_1^{65} \sqrt{u}\,\frac{du}{8}\, d\theta = \int_0^{2\pi}\left(u^{3/2}\cdot\frac{1}{12}\Big|_{u=1}^{u=65}\right)d\theta =$$

$$= \boxed{\frac{\pi}{6}\left(65^{3/2}-1\right)}$$

! Warning: We note that while here we got $r\,dr\,d\theta$ when using polar coordinates, we would not need factor "r" if we parametrize surface by r,θ.

Lecture #20

Ex 4: Evaluate the surface integral $\iint_S y\,dS$, where S is given by
$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle$, $0 \le u \le 1$
$0 \le v \le \pi$.

$$\left.\begin{aligned}\vec{r}_u &= \langle \cos v, \sin v, 0\rangle \\ \vec{r}_v &= \langle -u\sin v, u\cos v, 1\rangle\end{aligned}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \sin v \cdot \vec{i} - \cos v \cdot \vec{j} + u \cdot \vec{k}$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{1+u^2}.$$

So: $\iint_S y\,dS = \iint_D u\sin v \cdot \sqrt{1+u^2}\,dA$, where $D = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le \pi\}$

$$= \int_0^1 u\sqrt{1+u^2}\,du \cdot \int_0^\pi \sin v\,dv = \int_1^2 \sqrt{w}\,\frac{dw}{2} \cdot \left(-\cos v\Big|_{v=0}^{v=\pi}\right) = w^{3/2} \cdot \frac{1}{3}\Big|_{w=1}^{w=2} \cdot 2 =$$

$$= \boxed{\frac{2}{3}(2\sqrt{2} - 1)}$$

- **Oriented Surfaces**

To define surface integrals of vector fields, we need to restrict our attention only to a certain class of surfaces, called oriented.

At every point (x,y,z) on the surface S there are two unit normal vectors (orthogonal to the tangent plane to S at (x,y,z)).

Def: If it is possible to choose a unit normal vector $\vec{n}$ at every point $(x,y,z) \in S$, so that $\vec{n}$ varies continuously over S, then S is called an oriented surface and the given choice of $\vec{n}$ provides S with an orientation. (note: there are two orientations for oriented surfaces)

Basic example 1: S is given as a graph of function $z = f(x,y)$. Then S is naturally oriented with an upward orientation given by $\boxed{\vec{n} = \dfrac{-f_x \cdot \vec{i} - f_y \cdot \vec{j} + \vec{k}}{\sqrt{1 + f_x^2 + f_y^2}}}$ – see p. 4.

Basic example 2: If S is a smooth orientable surface given by $\vec{r}(u,v)$ then it has a canonical orientation given by

$$\boxed{\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

However: Möbius strip is one of the basic examples of a non-oriented surface.

• Surface Integrals of vector fields

Def: If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$ (defining the orientation), then the surface integral of F over S is

$$\boxed{\iint_S F\cdot dS = \iint_S \vec{F}\cdot\vec{n}\, dS}$$

This integral is also called the flux of F across S

In other words, if S is given by $\vec{r}(u,v)$ and we choose $\vec{n}$ as on p.5:

$$\iint_S \vec{F} dS = \iint_S \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}dS = \iint_D\left(\vec{F}(\vec{r}(u,v))\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\right)\cdot|\vec{r}_u\times\vec{r}_v|\,dA$$

$$= \boxed{\iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,dA}$$

Ex 5: Find the flux of the vector field $F(x,y,z) = \langle 3z, 3y, 3x\rangle$ across the sphere $S: x^2+y^2+z^2=R^2$.

► Choose the same parametrization as in Ex2.

$\vec{r}(\phi,\theta) = \langle R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi\rangle$.

Computed $\vec{r}_\phi\times\vec{r}_\theta = \langle R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi\rangle$.

while $\vec{F}(\vec{r}(\phi,\theta)) = \langle 3R\cos\phi, 3R\sin\phi\sin\theta, 3R\sin\phi\cos\theta\rangle$.

Hence: $\vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi\times\vec{r}_\theta) = 3R^3(\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta)$

So: $\iint_S \vec{F} dS = \int_0^{2\pi}\int_0^{\pi} 3R^3(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\,d\phi\,d\theta$

(1) $\int_0^{2\pi}\int_0^{\pi} 2\sin^2\phi\cos\phi\cos\theta\,d\phi\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta\cdot\int_0^{\pi}2\sin^2\phi\cos\phi\,d\phi = 0$ as $\int_0^{2\pi}\cos\theta\,d\theta = 0$.

(2) $\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\sin^2\theta\,d\phi\,d\theta = \int_0^{2\pi}\frac{1-\cos(2\theta)}{2}d\theta\cdot\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi = \pi\cdot\int_1^{-1}(1-u^2)\frac{du}{-1} = \frac{4}{3}\pi$

So: $\boxed{\iint_S \vec{F} dS = 3R^3\cdot\frac{4}{3}\pi = 4\pi R^3}$