Lecture #21 11 / 14 / 2017

- Last time: surface integrals

Ex 1: Evaluate the surface integral $\iint_S (2y^2+z^2)\,dS$, where S is the part of paraboloid $x=y^2+z^2$ given by $0\le x\le 1$.

- We start from parametrization of S:

$$\boxed{\vec{r}(y,z)=\langle y^2+z^2, y, z\rangle, \quad D=\{(y,z)\mid 0\le y^2+z^2\le 1\} \text{ - unit disk.}}$$

- Next, compute $|\vec{r}_y\times\vec{r}_z|$

$$\left.\begin{array}{l}\vec{r}_y=\langle 2y,1,0\rangle\\ \vec{r}_z=\langle 2z,0,1\rangle\end{array}\right\} \Rightarrow \vec{r}_y\times\vec{r}_z=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2y & 1 & 0\\ 2z & 0 & 1\end{vmatrix}=\langle 1,-2y,-2z\rangle$$

N.B: If you remember the f-la for the graph of function, you don't need above computation.

So: $|\vec{r}_y\times\vec{r}_z|=\sqrt{1+4(y^2+z^2)}$

- Therefore, our surface integral equals:

$$\iint_D (2y^2+z^2)\sqrt{1+4(y^2+z^2)}\,dA.$$

Using polar coordinates $y=r\cos\theta$, $z=r\sin\theta$, we get:

$$\int_0^{2\pi}\int_0^1 (2r^2\cos^2\theta+r^2\sin^2\theta)\cdot\sqrt{1+4r^2}\,r\,dr\,d\theta=\int_0^{2\pi}\left[(2\cos^2\theta+\sin^2\theta)\cdot\int_0^1 r^3\sqrt{1+4r^2}\,dr\right]d\theta.$$

First, evaluate inner integral

$$\int_0^1 r^3\sqrt{1+4r^2}\,dr \overset{u=1+4r^2}{=\!=\!=} \int_1^5 \sqrt{u}\cdot\frac{u-1}{4}\cdot\frac{du}{8}=\frac{1}{32}\int_1^5 (u^{3/2}-u^{1/2})\,du=\frac{1}{32}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Big|_1^5=$$

$$=\frac{2}{32}\left(\frac{1}{5}\cdot 25\sqrt{5}-\frac{1}{5}-\frac{1}{3}\cdot 5\sqrt{5}+\frac{1}{3}\right)=\frac{1}{16}\left(\frac{10\sqrt{5}}{3}+\frac{2}{15}\right)=\frac{1}{8}\left(\frac{25\sqrt{5}+1}{15}\right)=\frac{25\sqrt{5}+1}{120}$$

Hence, we get

$$\int_0^{2\pi}(2\cos^2\theta+\sin^2\theta)\cdot\frac{25\sqrt{5}+1}{120}\,d\theta=\frac{25\sqrt{5}+1}{120}\int_0^{2\pi}\left(2\cdot\frac{1+\cos(2\theta)}{2}+\frac{1-\cos(2\theta)}{2}\right)d\theta=\boxed{\frac{\pi}{40}\cdot(25\sqrt{5}+1)}$$

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Ex2: Find the area of the part S of the paraboloid $y=x^2+z^2$ that lies within the cylinder $x^2+z^2=16$.

▸ Want to compute for two standard parametrizations of paraboloid.

1st parametrization

S as a graph of function in x,z is parametrized as follows

$$\boxed{\vec{r}(x,z)=\langle x, x^2+z^2, z\rangle, \quad D=\{(x,z)\mid 0\le x^2+z^2\le 16\}} \text{ — disk of radius 4 centered at the origin}$$

We know (or can compute right on spot): $|\vec{r}_x\times\vec{r}_z|=\sqrt{1+4x^2+4z^2}$.

Hence, we reduce to

$$\iint_D \sqrt{1+4x^2+4z^2}\,dA.$$

Using polar coordinates $x=r\cos\theta$, $y=r\sin\theta$, this integral becomes

$$\int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\,r\,dr\,d\theta \underset{u=1+4r^2}{=\!=\!=} \int_0^{2\pi}\int_1^{65}\sqrt{u}\,\frac{du}{8}\,d\theta = \int_0^{2\pi}\left(\frac{2}{3}\cdot\frac{1}{8}\cdot u^{3/2}\right)\Big|_{u=1}^{u=65} d\theta = \frac{\pi}{6}(65^{3/2}-1).$$

So: Area$(S)=\boxed{\frac{\pi}{6}(65^{3/2}-1)}$

2nd parametrization

Let us immediately parametrize S by r,θ as follows:

$$\boxed{\vec{r}(r,\theta)=\langle r\cos\theta, r^2, r\sin\theta\rangle \quad D=\{(r,\theta)\mid 0\le r\le 4,\ 0\le\theta\le 2\pi\}}$$

$$\left.\begin{array}{l}\vec{r}_r=\langle\cos\theta, 2r, \sin\theta\rangle\\ \vec{r}_\theta=\langle -r\sin\theta, 0, r\cos\theta\rangle\end{array}\right\}\Rightarrow \vec{r}_r\times\vec{r}_\theta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos\theta & 2r & \sin\theta\\ -r\sin\theta & 0 & r\cos\theta\end{vmatrix}= 2r^2\cos\theta\cdot\vec{i}-r\vec{j}+2r^2\sin\theta\,\vec{k}$$

$$\Rightarrow |\vec{r}_r\times\vec{r}_\theta|=\sqrt{4r^4\cos^2\theta+r^2+4r^4\sin^2\theta}=r\sqrt{1+4r^2}.$$

So: we reduced to the same integral $\int_0^{2\pi}\int_0^4 r\sqrt{1+4r^2}\,dr\,d\theta$ and the rest of computations proceed as above.

! The key point that I wanted to emphasize in 2nd parametrization is that this extra factor "r" is already incorporated into $|\vec{r}_r\times\vec{r}_\theta|$, i.e., you do not integrate $r\sqrt{1+4r^2}\cdot r\,dr\,d\theta$, but rather $r\sqrt{1+4r^2}\,dr\,d\theta$ ∎

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In the end of last lecture we rushed through the definition of the surface integral of a vector field. Let's now remind formula and do a couple of examples.

Def: If $\vec{F}$ is a continuous vector field defined on an oriented surface S with orientation defined by unit vector $\vec{n}$ (at each point), then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F}\,dS := \iint_S \underbrace{\vec{F}\cdot\vec{n}}\,dS$$

this is now a function and we know how to integrate them.

The integral $\iint_S \vec{F}\,dS$ is called the flux of $\vec{F}$ across S.

The only nontrivial thing is what is $\vec{n}$. There are two basic examples we discussed last time.

1. Surface given as a graph of function $z = f(x,y)$

$$\vec{n} = \frac{-f_x\cdot\vec{i} - f_y\cdot\vec{j} + \vec{k}}{\sqrt{1+f_x^2+f_y^2}}$$ – the normal vector defining upward orientation

2. Smooth parametric surface $\vec{r}(u,v)$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$ – one of the two choices of the normal vector.

For a closed surface (i.e. boundary of the solid E), the positive orientation is the one where the normal vector points outside of E, while the inward-pointing normals give the negative orientation.

Example: S – sphere of radius R, centered at the origin

Last time: $|\vec{r}_\phi \times \vec{r}_\theta| = R^2 \sin\phi$

$\vec{r}_\phi \times \vec{r}_\theta = \langle R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi\rangle$

$$\Rightarrow \vec{n} = \frac{\vec{r}_\phi\times\vec{r}_\theta}{|\vec{r}_\phi\times\vec{r}_\theta|} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$$ – points in the same direction as the position vector $\vec{r}(\phi,\theta)$

clearly defines a positive orientation.

Ex3: Find the flux of the vector field $F(x,y,z) = \langle 3z, 3y, 3x\rangle$ across the sphere $S: x^2+y^2+z^2=R^2$.

$\vec{r}(\phi,\theta) = \langle R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi\rangle$

$\vec{r}_\phi \times \vec{r}_\theta = \langle R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi\rangle$

So: $\text{Flux}(F) = \int_0^{2\pi}\int_0^{\pi} \langle 3R\cos\phi, 3R\sin\phi\sin\theta, 3R\sin\phi\cos\theta\rangle \cdot \langle R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi\rangle \, d\phi\, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi} 3R^3(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\,d\phi\,d\theta.$$

(1) $\int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos\phi\cos\theta\,d\phi\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta \cdot \int_0^{\pi}\sin^2\phi\cos\phi\,d\phi = 0.$

(2) $\int_0^{2\pi}\int_0^{\pi} \sin^3\phi\sin^2\theta\,d\phi\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta \cdot \int_0^{\pi}\sin^3\phi\,d\phi = \int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta \cdot \int_0^{\pi}(1-\cos^2\phi)\frac{d(\cos\phi)}{-1} =$

$$= \pi \cdot \int_1^{-1}(1-u^2)\frac{du}{-1} = \pi\cdot\left(u - \frac{u^3}{3}\right)\Big|_{u=-1}^{u=1} = \frac{4}{3}\pi.$$

So: Flux of $\vec{F}$ across S equals $\boxed{4\pi R^3}$

Ex4: Find the flux of the vector field $\vec{F}(x,y,z) = \langle x,y,z\rangle$ over a cylinder $S: 0\le z\le 5,\ x^2+y^2=9$, where a positive orientation of S is chosen.

- First, we parametrize the cylinder $\vec{r}(\theta,t) = \langle 3\cos\theta, 3\sin\theta, t\rangle$

$D = \{(\theta,t)\mid 0\le\theta\le 2\pi,\ 0\le t\le 5\}.$

- Next, we compute $\vec{r}_\theta\times\vec{r}_t$

$$\left.\begin{array}{l}\vec{r}_\theta = \langle -3\sin\theta, 3\cos\theta, 0\rangle\\ \vec{r}_t = \langle 0, 0, 1\rangle\end{array}\right\} \Rightarrow \vec{r}_\theta\times\vec{r}_t = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -3\sin\theta & 3\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix} = \langle 3\cos\theta, 3\sin\theta, 0\rangle$$

discuss geom. meaning

! It is clear that $\langle 3\cos\theta, 3\sin\theta, 1\rangle$ points outwards.

- So: $\text{Flux}(F) = \iint_D \langle 3\cos\theta, 3\sin\theta, t\rangle\cdot\langle 3\cos\theta, 3\sin\theta, 0\rangle\,dA = \iint_D 9\,dA$

$$= \int_0^{2\pi}\int_0^5 9\,dt\,d\theta = \boxed{90\pi}$$

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Ex 5: Find the flux of the vector field $\vec{F}(x,y,z)=\langle x,y,z\rangle$ over a cylinder $S: 0\le z\le 5, x^2+y^2=9$ together with its "top" and bottom, i.e. we add disks $z=0, x^2+y^2\le 9$ and $z=5, x^2+y^2\le 9$. The orientation is positive.

▶ This is a generalization of the previous example.

Let $S_1: x^2+y^2\le 9, z=0$

$S_2: x^2+y^2\le 9, z=5$

Then, our surface is $S\sqcup S_1\sqcup S_2$ ⇒ we need to compute the flux of $\vec{F}$ along each of S, S_1, S_2 and then add them up.

- The flux of $\vec{F}$ across S was computed in Ex 4 and it is $\boxed{90\pi}$
- As S_1 is the "bottom", the positive orientation corresponds to $\vec{n}=-\vec{k}$ ⇒ flux of $\vec{F}$ across S_1 equals:

$$\iint_{x^2+y^2\le 9}\langle x,y,1\rangle\cdot\langle 0,0,-1\rangle\,dA=\boxed{0}$$

- As S_2 is the "top", the positive orientation corresponds to $\vec{n}=\vec{k}$. Hence, flux of $\vec{F}$ across S_2 equals

$$\iint_{x^2+y^2\le 9}\langle x,y,5\rangle\cdot\langle 0,0,1\rangle\,dA=\iint_{x^2+y^2\le 9}5\,dA=5\cdot\text{Area}\begin{pmatrix}\text{Disk of}\\ \text{radius }3\end{pmatrix}=\boxed{45\pi}$$

Thus, flux of $\vec{F}$ across $S\sqcup S_1\sqcup S_2$ is $\boxed{135\pi}$

∎

Ex 6: Let $\vec{F}(x,y,z)=\langle 0,y,-z\rangle$. Find the flux of $\vec{F}$ across the positively oriented S, which consists of the paraboloid $y=x^2+z^2, 0\le y\le 1$ and the disk $x^2+z^2\le 1, y=1$ on "top of it".

See the solution on the next page.

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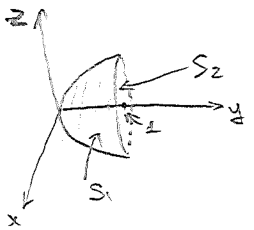
Solution of Ex6



This surface consists of two parts $S_1 \amalg S_2$, where:

S_1 – part of the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$

S_2 – disk $x^2 + z^2 \le 1$, $y = 1$.

For this reason, we evaluate $\iint_{S_1} \vec{F}\, dS$, $\iint_{S_2} \vec{F}\, dS$ separately and then add them up.

* Flux of $\vec{F}$ across S_2

S_2 is obviously parametrized by $\vec{r}(x,z) = \langle x, 1, z \rangle$, $D = \{(x,z) \mid x^2+z^2 \le 1\}$

There are two choices of orientation $\vec{n} = \vec{j}$ or $-\vec{j}$ and it is clear that the first looks outwards the enclosed solid $\Rightarrow$ choose $\vec{n} = \vec{j}$

$$\iint_{S_2} \vec{F}\, dS = \iint_D \langle 0, 1, -z \rangle \cdot \langle 0, 1, 0 \rangle\, dA = \iint_D 1\, dA = \text{Area}(D) = \boxed{\pi.}$$

* Flux of $\vec{F}$ across S_1

We parametrize S_1: $\vec{r}(x,z) = \langle x, x^2+z^2, z \rangle$, $D = \{(x,z) \mid x^2+z^2 \le 1\}$

$$\left.\begin{array}{l} \vec{r}_x = \langle 1, 2x, 0 \rangle \\ \vec{r}_z = \langle 0, 2z, 1 \rangle \end{array}\right\} \Rightarrow \vec{r}_x \times \vec{r}_z = \langle 2x, -1, 2z \rangle \leftarrow$$

it is clear that it is the normal vector looking outwards of the enclosed solid. This is seen by $-1 \cdot \vec{j}$ in it or you can pick any pt, e.g. $(0,0,0)$ and the above vector is $\langle 0,-1,0 \rangle$ which points outside

Therefore, the corresponding flux is:

$$\iint_D \langle 0, x^2+z^2, -z \rangle \cdot \langle 2x, -1, 2z \rangle\, dA = \iint_D (-x^2 - 3z^2)\, dA \underset{\substack{x = r\cos\theta \\ z = r\sin\theta}}{=} \int_0^{2\pi}\int_0^1 (-r^2\cos^2\theta - 3r^2\sin^2\theta) \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} (-\cos^2\theta - 3\sin^2\theta) \cdot \int_0^1 r^3\, dr\, d\theta = -\frac{1}{4} \cdot \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2} + 3 \cdot \frac{1-\cos(2\theta)}{2} \right) d\theta = \boxed{-\pi}$$

Thus: The flux of $\vec{F}$ across $S = S_1 \amalg S_2$ is $\pi + (-\pi) = \boxed{0} \leftarrow$ Next time we will see why this could be expected