Lecture #22 11/16/2017

- Today: Stokes' Theorem

This theorem can be viewed as a 3D-analogue of the Green's Theorem and it relates a surface integral over a surface S to a line integral around the boundary of S.

Convention: The orientation of S (given by a unit normal vector $\vec{n}$ at every point of S) induces the positive orientation of the boundary curve C. This means that if you walk in the positive direction around C with your head pointing in the direction of $\vec{n}$, then the surface is always on your left.

Stokes' Theorem: Let S be an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth boundary curve C (also denoted ∂S). Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then:

$$\boxed{\int_C \vec{F}\,dr = \iint_S \operatorname{curl}(\vec{F})\,dS}$$

This can be also written in the form $\boxed{\int_{\partial S} \vec{F}\,dr = \iint_S (\nabla \times \vec{F})\,dS}$

Toy example: If S lies in the xy-plane with an upward orientation, then we recover

$$\int_C \vec{F}\,dr = \iint_S \operatorname{curl}(\vec{F})\,dS = \iint_S \operatorname{curl}(\vec{F})\cdot\vec{k}\,dA$$

which is the vector form of Green's theorem we got before.

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Stokes' Theorem can be used in two directions:

(1) reduce computation of the surface integral of $\vec{F}$ across S to a computation of the line integral of $\vec{G}$ (if it exists) along ∂S, where $\vec{G}$ is an "uncurl" of $\vec{F}$, i.e. $\vec{F} = \text{curl}(\vec{G})$.

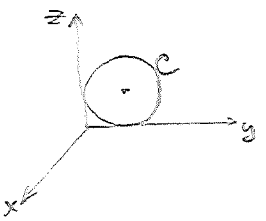
(Recall: such $\vec{G}$ exists iff $\text{div}\,\vec{F} = 0$)

(2) reduce computation of the line integral of $\vec{F}$ along the closed curve C to a computation of the surface integral of $\text{curl}(\vec{F})$ across any surface S whose boundary is C.

Ex1: Let C be the curve defined by the parametric equations

$$C: x = 0, \; y = 2 + 2\cos t, \; z = 2 + 2\sin t, \; 0 \le t \le 2\pi.$$

Evaluate $\int_C x^2 e^{5z}\,dx + x\cos y\,dy + 3y\,dz$



C – the circle of radius 2 centered at pt $(0,2,2)$ and entirely lying the yz–plane.

Choose S to be the disk in yz–plane bounded by C.

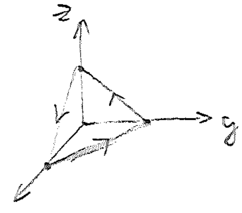
Stokes' Theorem: $\int_C x^2 e^{5z}\,dx + x\cos y\,dy + 3y\,dz = \iint_S \text{curl}(x^2 e^{5z}\vec{i} + x\cos y\,\vec{j} + 3y\vec{k})\,dS$

$$\text{curl}(x^2 e^{5z}\vec{i} + x\cos y\,\vec{j} + 3y\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 e^{5z} & x\cos y & 3y \end{vmatrix} = 3\vec{i} + 5x^2 e^{5z}\vec{j} + \cos y \cdot \vec{k}$$

So: $\iint_S \text{curl}(x^2 e^{5z}\vec{i} + x\cos y\,\vec{j} + 3y\vec{k})\,dS = \iint_S 3\,dA = 3 \cdot \text{Area}(S) = \boxed{12\pi}$

used that normal vector is obviously $\vec{i}$

Ex2: Evaluate $\int_C \vec{F}\,d\vec{r}$, where $\vec{F} = \langle z^2, y^2, x\rangle$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ with a counter-clockwise rotation.



$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = -\vec{j}(1-2z) = (2z-1)\vec{j}$$

To apply Stokes' Theorem we choose the surface S to be the triangle bounded by C with an upward orientation

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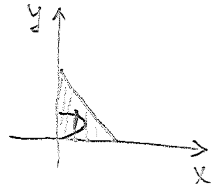
(Continuation of Ex2)

The equation of the plane containing our triangle is $x+y+z=1 \Leftrightarrow z=1-x-y$ and therefore it is parametrized via $\vec{r}(x,y)=(x,y,1-x-y)$, $D=\{(x,y)\mid \begin{smallmatrix} x,y\geq 0 \\ x+y\leq 1\end{smallmatrix}\}$

Then: $\int_C \vec{F}\,d\vec{r} = \iint_S \text{curl}\,\vec{F}\,d\vec{S} = \iint_S (2z-1)\vec{j}\,d\vec{S} \ominus$

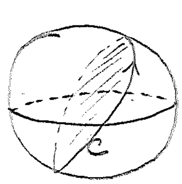
But we know that $\vec{r}_x \times \vec{r}_y = \langle 1,1,1\rangle$ (you can compute directly or use general formula we had for a graph of a function)

$\ominus \iint_D (2z-1)\vec{j}\cdot(\vec{i}+\vec{j}+\vec{k})\,dA = \iint_D (2z-1)\,dA = \iint_D (1-2x-2y)\,dA$, where D is the projection of S onto the xy-plane described above.



$$\iint_D (1-2x-2y)\,dA = \int_0^1\int_0^{1-x}(1-2x-2y)\,dy\,dx = \int_0^1\left[(1-2x)(1-x) - y^2\Big|_{y=0}^{y=1-x}\right]dx$$
$$= \int_0^1 (x^2-x)\,dx = \left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_{x=0}^{x=1} = \boxed{-\frac{1}{6}}$$

Ex3: Evaluate $\int_C \vec{F}\,d\vec{r}$ where $\vec{F}=\langle z-y,-x-z,-x-y\rangle$ and C is the curve $x^2+y^2+z^2=4$, $y=z$, oriented counterclockwise when viewed from above.



First of all we need to choose any surface with boundary C. The simplest choice is just to take the disk enclosed by C, i.e. $S=\{(x,y,z)\mid x^2+y^2+z^2\leq 4, y=z\}$.

$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & -x-z & -x-y \end{vmatrix} = 2\vec{j}$$

Thus, by Stokes' Theorem: $\int_C \vec{F}\,d\vec{r} = \iint_S 2\vec{j}\,d\vec{S}$

Next, we parametrize S: $\vec{r}(x,y)=\langle x,y,y\rangle$, where $(x,y)\in D := \{(x,y)\mid x^2+2y^2\leq 4\}$.

Then $\vec{r}_x\times\vec{r}_y = \langle 0,-1,1\rangle$

Hence: $\iint_S 2\vec{j}\,d\vec{S} = \iint_D \langle 0,2,0\rangle\cdot\langle 0,-1,1\rangle\,dA = \iint_D -2\,dA = -2\cdot\text{Area}(D) = -2\cdot\pi\cdot 2\cdot\sqrt{2} = \boxed{-4\sqrt{2}\pi}$,

where we used the formula for the area of an ellipse derived before.

N.B.: You could also compute $\iint_S 2\vec{j}\,d\vec{S}$ without explicit parametrization just by noticing that S lies in the plane $z-y=0 \Rightarrow \frac{1}{\sqrt{2}}(-\vec{j}+\vec{k}) =: \vec{n}$ – an upward normal vector

so that $\iint_S 2\vec{j}\,d\vec{S} = \iint_S 2\vec{j}\cdot\frac{-\vec{j}+\vec{k}}{\sqrt{2}}\,dS = -\sqrt{2}\cdot\text{Area}(S) = \boxed{-4\sqrt{2}\pi}$

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Now we use Stokes' Theorem in the opposite direction.

Ex 4: Evaluate $\iint_S \operatorname{curl} \vec{F}\, dS$, where $\vec{F} = \langle y^2 z, xz, x^2y^2 \rangle$ and S is the part of the paraboloid $z = x^2+y^2$ that lies in the cylinder $x^2+y^2=1$, oriented upwards.

By the Stokes' Theorem: $\iint_S \operatorname{curl} \vec{F}\, dS = \int_{\partial S} \vec{F}\, dr$

The boundary ∂S of S is unit circle in the plane $z=1$ parametrized as:
$\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$, $0 \le t \le 2\pi$.

In particular, $\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$
$\vec{F}(\vec{r}(t)) = \langle \sin^2 t, \cos t, \cos^2 t \sin^2 t \rangle$
$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\sin^3 t + \cos^2 t$

So: $\iint_S \operatorname{curl} \vec{F}\, dS = \int_{\partial S} \vec{F}\, dr = \int_0^{2\pi} (-\sin^3 t + \cos^2 t)\, dt = \boxed{\pi}$

$\int_0^{2\pi} \sin^3 t\, dt = \int_0^{2\pi} (1-\cos^2 t) \sin t\, dt \overset{u=\cos t}{=} \int_1^1 (1-u^2)(-du) = 0$

$\int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt = \pi$

Ex 5: Evaluate $\iint_S \operatorname{curl} \vec{F}\, dS$, where $\vec{F} = z^2 \vec{i} - 3xy \vec{j} + x^3y^3 \vec{k}$ and S is the part of paraboloid $z = 5 - x^2 - y^2$ above the plane $z=1$, oriented upwards

By Stokes' Theorem: $\iint_S \operatorname{curl} \vec{F}\, dS = \int_{\partial S} \vec{F}\, dr$

The boundary ∂S of S is the circle of radius 2 in the plane $z=1$ oriented counterclockwise: $\vec{r}(t) = \langle 2\cos t, 2\sin t, 1 \rangle$, $0 \le t \le 2\pi$.

Then: $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$
$\vec{F}(\vec{r}(t)) = \langle 1, -12\cos t \sin t, 64\cos^3 t \sin^3 t \rangle$
$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -2\sin t - 24\cos^2 t \sin t$

Thus: $\int_{\partial S} \vec{F}\, dr = \int_0^{2\pi} (-2\sin t - 24\cos^2 t \sin t)\, dt = \int_0^{2\pi} -24\cos^2 t \sin t\, dt \overset{u=\cos t}{=} 24 \int_1^1 u^2\, du = \boxed{0}$

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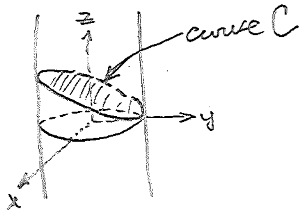
Ex6: Evaluate $\int_C \vec{F}\, d\vec{r}$, where $\vec{F} = \langle y^2, -x, -z^2 \rangle$ and C is the curve of intersection of the plane $y+2z=1$ and the cylinder $x^2+y^2=1$ oriented counterclockwise when viewed from above.



We apply Stokes' Theorem by choosing S to be the interior of the curve inside the given plane (so that S is an ellipse).

- First, let us compute $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & -x & -z^2 \end{vmatrix} = \vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}(-1-2y) = -(2y+1)\vec{k}$

- Next we parametrize S by viewing it as a graph of function $1-2y$, i.e. $S: \vec{r}(x,y) = \langle x, y, 1-2y \rangle$ and $(x,y) \in D$ - the unit disk $\{(x,y) \mid x^2+y^2 \le 1\}$. In particular, we know that from general formula $\vec{r}_x \times \vec{r}_y = \langle 0, 2, 1 \rangle$ and we need to pick one of $\pm\langle 0,2,1\rangle$ to be compatible with orientation of C. Easy to see it is $\langle 0,2,1 \rangle$.

Finally, we have:

$$\oint_C \vec{F}\, d\vec{r} \overset{\text{Stokes}}{=} \iint_S \operatorname{curl}\vec{F}\, d\vec{S} = \iint_S \langle 0,0,-2y-1\rangle\, d\vec{S} = \iint_D \langle 0,0,-2y-1\rangle \cdot \langle 0,2,1\rangle\, dA$$

$$= \iint_D (-2y-1)\, dA \xrightarrow[y = r\sin\theta]{x = r\cos\theta} \int_0^{2\pi}\int_0^1 (-2r\sin\theta - 1)\, r\, dr\, d\theta = \int_0^{2\pi} \left(-\frac{2}{3}\sin\theta - \frac{1}{2}\right) d\theta = -\pi$$

Ex7: Evaluate the integral $\iint_S \vec{F}\,dS$, where

$\vec{F} = \langle e^{y^2}, e^{x^4}\cos(z^3), y^2x\rangle$

S – part of the paraboloid $z = x^2+y^2$ below $z=4$ oriented upwards.

▸ If we are to apply the Stokes' Theorem on the nose, we would need to find a vector field $\vec{G}$ which uncurls $\vec{F}$, i.e. $\vec{F} = \mathrm{curl}(\vec{G})$.
The problem, however, is to find an explicit formula for $\vec{G}$

! However: the existence of such $\vec{G}$ is guaranteed by $\mathrm{div}(\vec{F})=0$.
Then, whatever $\vec{G}$ looks like, we have

$$\iint_S \vec{F}\,dS = \iint_S \mathrm{curl}(\vec{G})\,dS = \int_{\partial S} \vec{G}\,dr = \iint_{S'} \mathrm{curl}(\vec{G})\,dS = \iint_{S'} \vec{F}\,dS,$$

for any other surface S' with the same boundary.

The boundary of S is a circle $\{(x,y,2) \mid x^2+y^2=4\}$, hence, the simplest choice for S' is the disk $S' = \{(x,y,2) \mid x^2+y^2 \leq 4\}$.
The normal vector to S' compatible with orientation is $\vec{k}$, so that

$$\iint_{S'} \vec{F}\,dS = \iint_{S'} \langle e^{y^2}, e^{x^4}\cos(z^3), y^2x\rangle \cdot \langle 0,0,1\rangle\,dS = \iint_{S'} y^2x\,dS \ominus$$

Parametrizing S': $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2\rangle$; $D = \{(r,\theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$.

$$\left.\begin{array}{l}\vec{r}_r = \langle \cos\theta, \sin\theta, 0\rangle \\ \vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle\end{array}\right\} \Rightarrow \vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = r\vec{k}$$

$$\ominus \int_0^{2\pi}\int_0^1 r^3\cos^2\theta\sin\theta \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{5}\cos^2\theta\sin\theta\,d\theta = \boxed{0}$$

N.B: Alternatively, we could parametrize S' by (x,y) and then switch to polar coordinates.
■

! This approach is similar to the way we used FTLI when we could not find potential by replacing the curve.