Lecture #23 11/28/2017

• Last time: Stokes' Theorem

Let S be an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth boundary curve ∂S, and let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then:

$$\boxed{\int_{\partial S} \vec{F} dr = \iint_S curl(F) dS}$$

Rmk 1: If S lies in the xy-plane with an upward orientation, then we recover the vector form of Green's Theorem we got before:

$$\boxed{\int_{C=\partial S} \vec{F} dr = \iint_S curl(\vec{F}) dS = \iint_S curl(\vec{F}) \cdot \vec{k}\, dA}$$

Rmk 2: If S' is another surface as above s.t. $\partial S = \partial S'$, then we get

$$\boxed{\iint_S curl(\vec{F}) dS = \iint_{S'} curl(\vec{F}) dS} \quad (*)$$

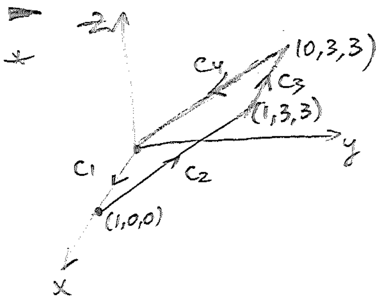
! Note that sometimes it is impossible to recover an explicit formula of the vector field $\vec{F}$ which "uncurls" a given vector field $\vec{G}$, but to apply $(*)$ it suffices to know such $\vec{F}$ exists. The latter is guaranteed (under same assumptions of continuous derivatives) by $\operatorname{div} \vec{G} = 0$

Rmk 3: As we learnt last time, Stokes' Theorem can be used 2 ways:

- to reduce a computation of a line integral to a computation of the surface integral, which is quite easier in many situations (surprisingly!)
- to reduce a computation of a surface integral of a vector field $curl(\vec{F})$ to a computation of a line integral of $\vec{F}$

Lecture #23

Ex1: Evaluate $\int_C \vec{F}\,d\vec{r}$, where $\vec{F} = x^2\vec{i} + 4xy^3\vec{j} + y^2x\vec{k}$ and C is the boundary of the rectangle with vertices $(0,0,0), (1,0,0), (1,3,3), (0,3,3)$ walked around in this order. Compute via Stokes' Theorem and compare to a direct computation.



$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 4xy^3 & y^2x \end{vmatrix} = 2xy\cdot\vec{i} - y^2\vec{j} + 4y^3\cdot\vec{k}$$

Choose S to be the "insides" of this rectangle. It has a natural parametrization

$$S: \vec{r}(x,y) = \langle x, y, y\rangle, \quad D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 3\}$$

graph of a function $z = f(x,y) = y$.

Therefore, $\vec{n} = -\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k} = -\vec{j} + \vec{k}$.

So: $\int_C \vec{F}\,d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot\vec{n}\,dS = \iint_D \langle 2xy, -y^2, 4y^3\rangle\cdot\langle 0,-1,1\rangle\,dA = \iint_D (4y^3+y^2)\,dA$

$$= \int_0^1\int_0^3 (4y^3+y^2)\,dy\,dx = \int_0^1 \left(y^4 + \frac{y^3}{3}\right)\Big|_{y=0}^{y=3} dx = \int_0^1 90\,dx = \boxed{90.}$$

*Let us now compute $\int_C \vec{F}\,d\vec{r} = \int_C x^2dx + 4xy^3dy + y^2x\,dz$ directly.

Split C into 4 segments C_1, C_2, C_3, C_4.

C_1: $\vec{r}(t) = \langle t, 0, 0\rangle$, $0 \le t \le 1$

$$\int_{C_1} x^2dx + 4xy^3dy + y^2x\,dz = \int_0^1 t^2\,dt = \frac{1}{3}$$

C_2: $\vec{r}(t) = \langle 1, t, t\rangle$, $0 \le t \le 3$

$$\int_{C_2} x^2dx + 4xy^3dy + y^2x\,dz = \int_0^3 (4t^3+t^2)\,dt = \left(t^4 + \frac{t^3}{3}\right)\Big|_{t=0}^{t=3} = 90$$

C_3: $\vec{r}(t) = \langle 1-t, 3, 3\rangle$, $0 \le t \le 1$

$$\int_{C_3} x^2dx + 4xy^3dy + y^2x\,dz = \int_0^1 -(1-t)^2\,dt = -\frac{1}{3}$$

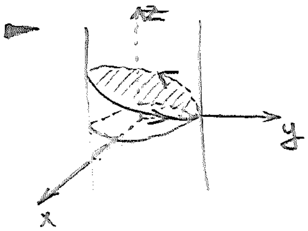
C_4: $\vec{r}(t) = \langle 0, 3-t, 3-t\rangle$, $0 \le t \le 3$

$$\int_{C_4} x^2dx + 4xy^3dy + y^2x\,dz = \int_0^3 0\,dt = 0$$

So: $\int_C \vec{F}\,d\vec{r} = \int_{C_1}\vec{F}\,d\vec{r} + \int_{C_2}\vec{F}\,d\vec{r} + \int_{C_3}\vec{F}\,d\vec{r} + \int_{C_4}\vec{F}\,d\vec{r} = \frac{1}{3} + 90 - \frac{1}{3} + 0 = \boxed{90}$

Lecture #23

Ex2: Evaluate $\int_C \vec{F}\, d\vec{r}$, where $\vec{F} = \langle y^2, -x, -z^2 \rangle$ and C is the curve of intersection of the plane $y+2z=1$ and the cylinder $x^2+y^2=1$ oriented counterclockwise when viewed from above.



Choose S to be the interior of the corresponding curve C.

- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & -x & -z^2 \end{vmatrix} = (-2y-1)\vec{k}$

- Parametrize S: $\vec{r}(x,y) = \langle x, y, \frac{1-y}{2} \rangle$, $D = \{(x,y) \mid x^2+y^2 \le 1\}$.

Then $\vec{n} = -\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k} = \frac{1}{2}\vec{j} + \vec{k}$ – defines an upward orientation of S compatible with the orientation of C.

So: $\int_C \vec{F}\, d\vec{r} = \iint_S (\operatorname{curl} \vec{F})\, d\vec{S} = \iint_D \langle 0, 0, -2y-1 \rangle \cdot \langle 0, \frac{1}{2}, 1 \rangle\, dA = \iint_D (-2y-1)\, dA$

$$= \int_0^{2\pi}\int_0^1 (-2r\sin\theta - 1)\, r\, dr\, d\theta = \int_0^{2\pi} \left(-\frac{2}{3}\sin\theta - \frac{1}{2}\right) d\theta = \boxed{-\pi}$$

Ex3: Evaluate $\int_C \vec{F}\, d\vec{r}$, where $\vec{F} = \langle y^2, 2x^2, e^{z^2} \rangle$ and C is the curve of intersection of the hemisphere $\{x^2+y^2+z^2=25,\ z \ge 0\}$ and the cylinder $x^2+y^2=9$, oriented counterclockwise when viewed from above

The key observation is that C is just a circle: $\{(x,y,4) \mid x^2+y^2=9\}$ and hence for S we can pick up its "interior", i.e. $S: \vec{r}(x,y) = \langle x, y, 4 \rangle$, $D = \{(x,y) \mid x^2+y^2 \le 9\}$.

- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & 2x^2 & e^{z^2} \end{vmatrix} = (4x-2y)\vec{k}$

- $\vec{n} = \vec{k}$ on S

So: $\int_C \vec{F}\, d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\, d\vec{S} = \iint_D \langle 0, 0, 4x-2y \rangle \cdot \langle 0, 0, 1 \rangle\, dA = \int_0^{2\pi}\int_0^3 (4r\cos\theta - 2r\sin\theta)\, r\, dr\, d\theta = 0.$

Lecture #23

Ex 4: Evaluate the integral $\iint_S \vec{F}\,d\vec{S}$, where $\vec{F} = \langle e^{y^2}, e^{x^4}\cos(z^3), y^2x^2 \rangle$ and S – the part of the paraboloid $z = x^2+y^2$ below $z=4$ oriented upwards.

If we are to apply Stokes' Theorem, then we would need to find a vector field $\vec{G}$ s.t. $\text{curl}(\vec{G}) = \langle e^{y^2}, e^{x^4}\cos(z^3), y^2x^2 \rangle$. This is impossible to do in a straightforward way!

However, the existence of such $\vec{G}$ is guaranteed by $\text{div}\,\vec{F} = 0$. Therefore, applying (*) from page 1, we see that

$$\iint_S \vec{F}\,d\vec{S} = \iint_{S'} \vec{F}\,d\vec{S},$$

where S': $\vec{r}(x,y) = \langle x, y, 2 \rangle$, $D = \{(x,y) \mid x^2+y^2 \le 4\}$.

Note that the normal vector to S' is just $\vec{k}$ (we choose $\vec{k}$ and not $-\vec{k}$ since S was oriented upwards).

So: $\iint_{S'} \vec{F}\,d\vec{S} = \iint_{S'} \langle e^{y^2}, e^{x^4}\cos(z^3), y^2x^2 \rangle \cdot \langle 0,0,1 \rangle\, dS = \iint_{S'} y^2x^2\, dS = \iint_D x^2y^2\, dA$

$$= \int_0^{2\pi}\int_0^2 r^4 \sin^2\theta\cos^2\theta \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{32}{6} \cdot \sin^2\theta\cos^2\theta\, d\theta = \int_0^{2\pi} \frac{32}{6} \cdot \frac{\sin^2 2\theta}{4}\, d\theta =$$

$$= \int_0^{2\pi} \frac{32}{6\cdot 4} \cdot \frac{1-\cos(4\theta)}{2}\, d\theta = \boxed{\frac{4}{3}\pi}$$

Rmk: Among the two extra problems on the current homework, one is similar to Ex 4, while the other requires you to actually find an explicit formula for uncurling vector field and then evaluate the corresponding line integral.