Lecture #24 11/30/2017

- New topic: Triple integrals

Today and on Tue we will consider triple integrals, which are pretty similar to double integrals, but a bit more complicated.

Want: Integrate a function $f(x,y,z)$ over a solid $E \subset \mathbb{R}^3$.

As in the case of double integrals, we will start from the simplest case

$$\boxed{B=[a,b]\times[c,d]\times[r,s] = \{(x,y,z) \mid a\le x\le b,\ c\le y\le d,\ r\le z\le s\}}$$

rectangular box

As in the case of usual integrals and double integrals, we define $\iiint\limits_B f(x,y,z)\,dV$ as a limit of the corresponding triple Riemann sums.

We will not need an actual precise definition, but will only need the practical method for their evaluation by reducing to iterated integrals.

Theorem (Fubini's Theorem for triple integrals): If $f(\cdot,\cdot,\cdot)$ is continuous on the rectangular box $B=[a,b]\times[c,d]\times[r,s]$, then:

$$\boxed{\iiint\limits_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz}$$

! Note that there are 6 possibilities for the RHS depending on the order of dx, dy, dz in the iterated integral

Ex 1: Evaluate the triple integral $\iiint\limits_B 6xye^z\,dV$, where

$$B=\{(x,y,z) \mid -1\le x\le 2,\ 0\le y\le 2,\ 0\le z\le 1\}.$$

In this problem we can chose any order (out of 6 possible).

$$\iiint\limits_B 6xye^z\,dV = \int_0^1\int_0^2\int_{-1}^2 6xye^z\,dx\,dy\,dz = \int_0^1\int_0^2 3ye^z\cdot x^2\Big|_{x=-1}^{x=2}\,dy\,dz = \int_0^1 9e^z\cdot\frac{y^2}{2}\Big|_{y=0}^{y=2}\,dz$$

$$= \int_0^1 18e^z\,dz = \boxed{18(e-1)}$$

Lecture #24

Similarly to the double integrals, we can also define the triple integral of $f(\cdot,\cdot,\cdot)$ over a general bounded region E in $\mathbb{R}^3$. Namely, pick a rectangular box $B \subset \mathbb{R}^3$ containing E and define function F on B:

$$F(x,y,z) = \begin{cases} f(x,y,z), & \text{if } (x,y,z) \in E \\ 0, & \text{if } (x,y,z) \in B \setminus E \end{cases}$$

Then we define

$$\boxed{\iiint\limits_E f(x,y,z)\,dV = \iiint\limits_B F(x,y,z)\,dV.}$$

On the technical side, this means that as we represent this triple integral by an iterated integral, the limits of integration are no longer fixed.

<u>Example</u>: If $E = \{(x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}$, then

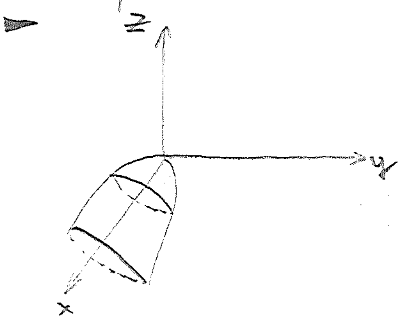
$$\iiint\limits_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

<u>Example</u>: If $E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$, then

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \Big[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\Big]\,dA.$$

(! Here D is just a projection of E on yz-plane)

<u>Ex2</u>: Evaluate $\iiint\limits_E \sqrt{y^2+z^2}\,dV$, where E is the region bounded by the paraboloid $x = y^2+z^2$ and the plane $x = 9$



Similarly to double integrals it is important to decide what should be the inner integration w.r.t.

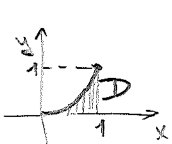
Let us take inner integration to be w.r.t. x, while using second Example from above, we set D to be the projection of E onto yz-plane, i.e. $D = \{(y,z) \mid y^2+z^2 \le 9\}$.

<u>Then</u>: $$\iiint\limits_E \sqrt{y^2+z^2}\,dV = \iint\limits_D \Big[\int_{y^2+z^2}^{9} \sqrt{y^2+z^2}\,dx\Big]\,dA = \iint\limits_D (9-y^2-z^2)\sqrt{y^2+z^2}\,dA \underset{z = r\sin\theta}{\overset{y = r\cos\theta}{=\!=\!=}}$$

$$= \int_0^{2\pi}\int_0^3 (9-r^2)\cdot r \cdot r\,dr\,d\theta = \int_0^{2\pi} \Big(\frac{9r^3}{3} - \frac{r^5}{5}\Big)\Big|_{r=0}^{r=3}\,d\theta = \Big(81 - \frac{243}{5}\Big)\cdot 2\pi = \boxed{\frac{324}{5}\pi}$$

Lecture #24

Ex3: Evaluate the triple integral $\iiint_E xy\,dV$, where E lies under the plane $z=2+x+y$ and above the region in the xy-plane bounded by $y=x^2$, $y=0$, $x=1$.

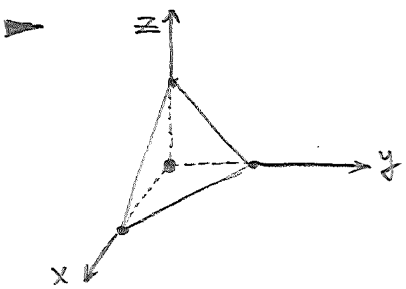


$$\iiint_E xy\,dV = \iint_D \left[\int_0^{2+x+y} xy\,dz\right] dA = \iint_D xy(2+x+y)\,dA = \int_0^1\int_0^{x^2} (2xy+x^2y+xy^2)\,dy\,dx$$

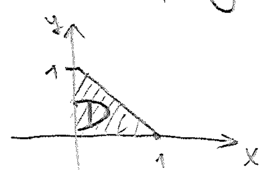
$$= \int_0^1 \left((2x+x^2)\cdot\frac{y^2}{2}\Big|_{y=0}^{y=x^2} + x\cdot\frac{y^3}{3}\Big|_{y=0}^{y=x^2} \right)dx = \int_0^1 \left(x^5+\frac{x^6}{2}+\frac{x^7}{3}\right)dx =$$

$$= \left(\frac{x^6}{6}+\frac{x^7}{14}+\frac{x^8}{24}\right)\Big|_{x=0}^{x=1} = \boxed{\frac{1}{6}+\frac{1}{14}+\frac{1}{24}}$$

Ex 4: Evaluate $\iiint_E 6xy\,dV$, where E is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.



Let the inner integration be w.r.t. z.
Then $D :=$ projection of E onto xy-plane



So: $$\iiint_E 6xy\,dV = \iint_D \left[\int_0^{1-x-y} 6xy\,dz\right] dA = \iint_D 6xy(1-x-y)\,dA =$$

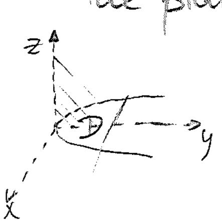
$$= \int_0^1\int_0^{1-x} (6xy-6x^2y-6xy^2)\,dy\,dx = \int_0^1 \left[(6x-6x^2)\cdot\frac{y^2}{2}\Big|_0^{1-x} - 6x\cdot\frac{y^3}{3}\Big|_{y=0}^{y=1-x}\right]dx =$$

$$= \int_0^1 \left[(3x-3x^2)(1-2x+x^2) - 2x\cdot(1-3x+3x^2-x^3)\right]dx =$$

$$= \int_0^1 (3x-6x^2+3x^3-3x^2+6x^3-3x^4-2x+6x^2-6x^3+2x^4)\,dx$$

$$= \int_0^1 (x-3x^2+3x^3-x^4)\,dx = \boxed{\frac{1}{2}-1+\frac{3}{4}-\frac{1}{5}}$$

Ex 5: Find the volume of the solid E enclosed by the cylinder $y=x^2$ and the planes $z=0$ and $y+z=1$.



$$Vol(E) = \iiint_E 1\,dV = \iint_D \left[\int_0^{1-y} 1\,dz\right] dA = \iint_D (1-y)\,dA = \int_{-1}^1\int_{x^2}^1 (1-y)\,dy\,dx =$$

$$= \int_{-1}^1 \left(y-\frac{y^2}{2}\right)\Big|_{y=x^2}^{y=1} dx = \int_{-1}^1 \left(\frac{1}{2}-x^2+\frac{x^4}{2}\right)dx = \left(\frac{1}{2}x-\frac{x^3}{3}+\frac{x^5}{10}\right)\Big|_{x=-1}^{x=1} = 1-\frac{2}{3}+\frac{1}{5} = \boxed{\frac{8}{15}}$$

Lecture #24

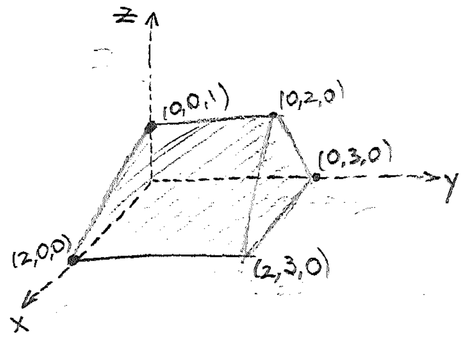
The previous example is based on a simple but useful formula:

$$\boxed{Vol(E) = \iiint_E 1\,dV}$$

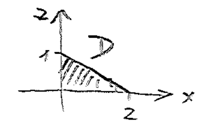
Rmk: In the simplest case, when $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$, then $\iiint_E 1\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} 1\,dz\right] dA = \iint_D [u_2(x,y) - u_1(x,y)]\,dA$ and we know that the latter equals $Vol(E)$.

Ex6: Describe the solid whose volume is given by the iterated integral

$$\int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{3-z} dy\,dz\,dx$$



First, we recover $D \subset \mathbb{R}^2_{xz}$ given by $\{(x,z) \mid 0 \leq x \leq 2,\ 0 \leq z \leq 1 - \frac{x}{2}\}$



Next, inner integral tells us $0 \leq y \leq 3 - z$
$\rightsquigarrow$ our solid is E lies under the plane $y = 3 - z$ and over the region D in xz-plane ∎

Another application of triple integrals is to the mass and center of mass. Explicitly, if $\rho(x,y,z)$ is the density function of a solid object occupying region E, then its mass equals

$$\boxed{m = \iiint_E \rho(x,y,z)\,dV}$$

while the center of mass is located at the point:

$$\boxed{\left(\frac{1}{m}\iiint_E x\rho(x,y,z)\,dV,\ \frac{1}{m}\iiint_E y\rho(x,y,z)\,dV,\ \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV\right)}$$

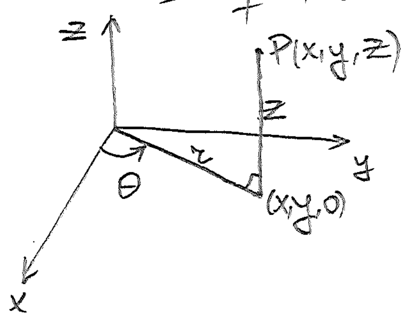
Finally, the total electric charge on a solid object occupying a region E and having charge density $\sigma(x,y,z)$ is

$$\boxed{Q = \iiint_E \sigma(x,y,z)\,dV}$$

Lecture #24

• Cylindrical Coordinates

Analogously to polar coordinates in $\mathbb{R}^2$, one can work with the cylindrical coordinate system in $\mathbb{R}^3$, where a point P is represented by (r, θ, z), where z is the usual z-coordinate, while (r, θ) are polar coordinates of the projection of P on the xy-plane.



To convert from cylindrical to rectangular:

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta, \quad z = z}$$

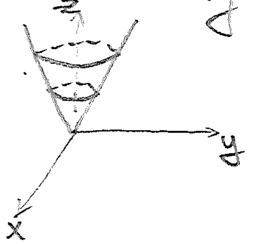
while to convert from rectangular to cylindrical

$$\boxed{r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z}$$

Ex 7: Describe the surface whose equation in cylindrical coordinates is $z = 2r$.

▸ $z = 2r = 2\sqrt{x^2+y^2}$, while θ-any.

This obviously determines a cone



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Ex 8: Describe the surface whose equation is $r^2 + z^2 = 9$.

▸ $\left.\begin{array}{l} 9 = r^2 + z^2 = x^2 + y^2 + z^2 \\ \theta\text{-any} \end{array}\right\} \Rightarrow$ a sphere of radius 3 centered at the origin.

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Lecture #24

- Triple Integrals in Cylindrical coordinates

Suppose that $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$ and D is conveniently described in polar coordinates, i.e. $D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$. Then, from the equality

$$\iiint_E f(x,y,z)\,dV = \iint_D \Big[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\Big]\,dA$$

switching $(x,y) \rightsquigarrow (r,\theta)$ we get:

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\cdot r\,dz\,dr\,d\theta}$$

Formula for triple integration in cylindrical coordinates.

Ex 9: Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region inside the cylinder $x^2+y^2=4$ and between the planes $z=-1$ and $z=2$.

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^2\int_{-1}^2 r^2\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 3r^2\,dr\,d\theta = \int_0^{2\pi} r^3\Big|_{r=0}^{r=2} d\theta = 16\pi$$

Ex 10: Find the volume of the solid E enclosed ~~enclosed~~ by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=2$

The intersection of the cone and sphere from above is a circle $\{(x,y,1) \mid x^2+y^2=1\}$ $\Rightarrow$ projection D of E onto xy-plane is a unit disc.

$$\mathrm{Vol}(E) = \iiint_E 1\,dV = \iint_D \Big[\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} 1\,dz\Big]\,dA = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^1 \left(r\sqrt{2-r^2} - r^2\right)dr\,d\theta = 2\pi\cdot\left(\int_0^1 r\sqrt{2-r^2}\,dr - \frac{r^3}{3}\Big|_{r=0}^{r=1}\right)$$

$$u = 2-r^2 \Rightarrow \int_0^1 r\sqrt{2-r^2}\,dr = \int_2^1 \sqrt{u}\,\frac{du}{-2} = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{u=1}^{u=2} = \frac{1}{3}(2^{3/2}-1) = \frac{2\sqrt{2}-1}{3}$$

So: $\mathrm{Vol}(E) = 2\pi\cdot\left(\frac{2\sqrt{2}-1}{3} - \frac{1}{3}\right) = \frac{4\pi}{3}(\sqrt{2}-1)$