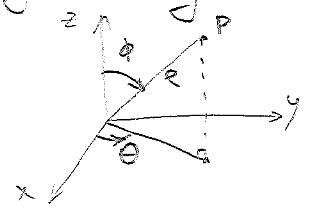
Lecture #25 12/05/2017

- Finish Ex9 & Ex10 from last class.
- Today: Triple integrals in spherical coordinates.

Recall that the spherical coordinates (ρ, θ, ϕ) of a point P in space are given by the following picture



Here $\rho \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

Ex1: (a) Describe points with $\theta = \theta_0 \leftarrow$ constant fixed angle.

(b) Describe points with $\phi = \phi_0 \leftarrow$ constant fixed angle.

▶ (a) half of the vertical plane, bounded by z-axis.

(b) half-cone, but for $\phi_0 = 0$ – positive part of ~~z~~-axis, $\phi_0 = \pi$ – negative part of z-axis

$\phi_0 = \pi/2$ – xy-plane ■

Recall: $\boxed{x = \rho \sin\phi \cos\theta, \; y = \rho \sin\phi \sin\theta, \; z = \rho\cos\phi}$

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

Ex2: (a) The point $(3, \underset{\theta}{\pi/3}, \underset{\phi}{\pi/4})$ is given in spherical coordinates. Find its rectangular coordinates

(b) The point $(3, 0, 4)$ is given in rectangular coordinates. Find its spherical coordinates

▶ (a) $x = 3 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{3\sqrt{2}}{4}$, $y = 3 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{6}}{4}$, $z = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} \Rightarrow \left(\frac{3\sqrt{2}}{4}, \frac{3\sqrt{6}}{4}, \frac{3\sqrt{2}}{2}\right)$

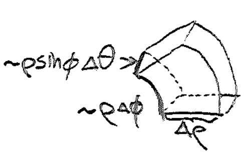
(b) Clearly $y = 0 \Rightarrow \theta = 0$, $\rho = \sqrt{3^2 + 4^2} = 5$. Finally $\cos\phi = \frac{z}{\rho} = \frac{4}{5} \Rightarrow \phi = \cos^{-1}\left(\frac{4}{5}\right)$. ■

- Evaluating triple integrals in (ρ, θ, ϕ)

Consider a counterpart of the rectangular box in spherical coordinates, aka a spherical wedge

$$\boxed{E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \; \alpha \leq \theta \leq \beta, \; c \leq \phi \leq d\}}$$

$a \geq 0$
$\beta - \alpha \leq 2\pi$
$d - c \leq \pi$



$\mathrm{Vol}(E) \approx (\Delta\rho) \cdot (\rho\Delta\phi) \cdot (\rho\sin\phi\,\Delta\theta) = \rho^2 \sin\phi \cdot \Delta\rho\Delta\phi\Delta\theta$

This implies the following formula for integration in spherical coordinates

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi, \text{ where } E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \; \alpha \leq \theta \leq \beta, \; c \leq \phi \leq d\}}$$

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Clearly the above formula (from the very end of p.1) can be generalized to spherical regions

$$E=\{(\rho,\theta,\phi)\mid \alpha\le\theta\le\beta,\ c\le\phi\le d,\ g_1(\theta,\phi)\le\rho\le g_2(\theta,\phi)\}$$

Ex 3: (a) Set up the integral $\iiint_B e^{\frac{1}{3}(x^2+y^2+z^2)^{3/2}}\,dV$, where $B=\{(x,y,z)\mid x^2+y^2+z^2\le 4\}$ in rectangular coordinates. Can you compute it?

(b) —//— in cylindrical coordinates. Can you compute it?

(c) —//— in spherical coordinates. Compute it!

► (a) $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} e^{\frac{1}{3}(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

(b) $$\int_0^{2\pi}\int_0^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} e^{\frac{1}{3}(r^2+z^2)^{3/2}}\,r\,dz\,dr\,d\theta$$

(c) $$\int_0^{\pi}\int_0^{2\pi}\int_0^{2} e^{\frac{1}{3}\rho^3}\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^{\pi}\int_0^{2\pi}\sin\phi\cdot\left(e^{\frac{1}{3}\rho^3}\Big|_{\rho=0}^{\rho=2}\right)d\theta\,d\phi=2\cdot 2\pi\cdot\left(e^{8/3}-1\right)=4\pi\left(e^{8/3}-1\right)$$ ■

Ex 4: Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{\frac{x^2+y^2}{3}}$ and below the sphere $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$.

► The eq-n of the sphere $\Leftrightarrow x^2+y^2+z^2=z \Leftrightarrow \rho^2=\rho\cos\theta \Leftrightarrow \rho=\cos\theta$.

The eq-n of the cone: $\rho\cos\phi=\sqrt{\frac{\rho^2\sin^2\phi}{3}}=\rho\cdot\frac{\sin\phi}{\sqrt{3}} \Leftrightarrow \tan(\phi)=\sqrt{3} \Leftrightarrow \phi=\pi/3$.

Hence: the solid is given in spherical coordinates by:

$$E=\{(\rho,\theta,\phi)\mid 0\le\theta\le 2\pi,\ 0\le\phi\le\pi/3,\ 0\le\rho\le\cos\phi\}$$

$$\mathrm{Vol}(E)=\iiint_E dV=\int_0^{\pi/3}\int_0^{2\pi}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\frac{2\pi}{3}\cdot\int_0^{\pi/3}\cos^3\phi\cdot\sin\phi\,d\phi=\frac{2\pi}{3}\cdot\int_1^{1/2}u^3\,du\cdot(-1)$$

$$=\frac{2\pi}{3}\cdot\frac{u^4}{4}\Big|_{u=1/2}^{u=1}=\frac{\pi}{6}\cdot\left(1-\frac{1}{16}\right)\cdot\frac{1}{4}=\frac{15\pi}{24\cdot 16}=\frac{5\pi}{128}$$ ■