- Last time: spherical coordinates

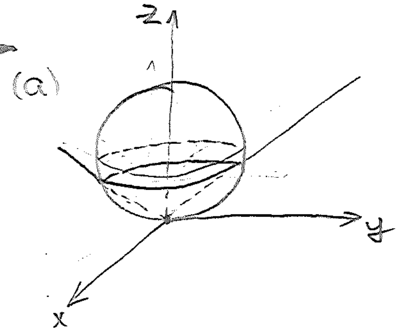
Ex1: Setup the integrals evaluating $\iiint_B f(x,y,z)\,dV$:

(a) B – "insides" ~~part~~ of the cone $z=\sqrt{\frac{x^2+y^2}{3}}$ bounded from above by the plane Π containing the curve of intersection of the above cone and sphere $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$. Use spherical coordinates and cylindrical coordinates.

(b) B – part of the ball $x^2+y^2+(z-\frac{1}{2})^2\le\frac{1}{4}$ lying above plane Π from (a). Setup both in spherical and cylindrical coordinates.

(c) B – part of the ball $x^2+y^2+(z-\frac{1}{2})^2\le\frac{1}{4}$ which is outside the cone $z=\sqrt{\frac{x^2+y^2}{3}}$. Setup in spherical and cylindrical coordinates.

(a)



Last time we computed $\phi=\frac{\pi}{3}$ – angle of the given cone. Let us now find the "height" of Π which is parallel to xy-plane.

$x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4} \iff x^2+y^2+z^2=z$

$z=\sqrt{\frac{x^2+y^2}{3}} \iff \begin{cases} x^2+y^2=3z^2 \\ z\ge 0\end{cases}$

$\Rightarrow$ for points of intersection $4z^2=z \Rightarrow z=\frac{1}{4}$.

So: $$\iiint_B f(x,y,z)\,dV=\int_0^{\pi/3}\int_0^{2\pi}\int_0^{1/4\cos\phi} f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

$$\iiint_B f(x,y,z)\,dV=\int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{r/\sqrt{3}}^{1/4} f(r\cos\theta,r\sin\theta,z)\cdot r\,dz\,dr\,d\theta.$$

(b) Analogous computations are based on the explicit value for value of ρ corresponding to the point of the given sphere with angle ϕ.

$$\iiint_B f(x,y,z)\,dV=\int_0^{\pi/3}\int_0^{2\pi}\int_{1/4\cos\phi}^{\cos\phi} f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$\iiint_B f(x,y,z)\,dV=\int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{1/4}^{\frac{1+\sqrt{1-4r^2}}{2}} f(r\cos\theta,r\sin\theta,z)\,r\,dz\,dr\,d\theta+\int_0^{2\pi}\int_{\sqrt{3}/4}^{1/2}\int_{\frac{1-\sqrt{1-4r^2}}{2}}^{\frac{1+\sqrt{1-4r^2}}{2}} -\!\!-\!\!\parallel\!\!-\!\!-$$

(c) $$\iiint_B f(x,y,z)\,dV=\int_{\pi/3}^{\pi/2}\int_0^{2\pi}\int_0^{\cos\phi} f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$\iiint_B f(x,y,z)\,dV=\int_0^{2\pi}\int_0^{\sqrt{3}/4}\int_{\frac{1-\sqrt{1-4r^2}}{2}}^{1/4} f(r\cos\theta,r\sin\theta,z)\,r\,dz\,dr\,d\theta.$$

Lecture #26

• **Today: Divergence Theorem**

Thm: Let E be a bounded closed solid region (in $\mathbb{R}^3$) with a piece-wise smooth boundary S endowed with an outward orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region that contains E. Then:

$$\boxed{\iint_S \vec{F}\, dS = \iiint_E \operatorname{div}(\vec{F})\, dV}$$

Ex2: Find the flux of the vector field $\vec{F} = \langle y+e^{z^2}, -\sin(x^3)+z^{10}, 3z - x^{20}y^{100}\rangle$ over the sphere $x^2+y^2+z^2=4$

▸ $\operatorname{div}(\vec{F}) = 0+0+3=3$. Clearly the above sphere S is a boundary of the ball $B: x^2+y^2+z^2 \le 4$ of radius 2. Hence:

$$\iint_S \vec{F} dS = \iiint_B \operatorname{div}(\vec{F})\, dV = 3\,\mathrm{Vol}(B) = 3\cdot\frac{4}{3}\pi\cdot 2^3 = \boxed{32\pi}$$

∎

Ex3: Prove that $\iint_S \operatorname{curl} \vec{F}\, dS = 0$ for $S, \vec{F}$ as in above Thm.

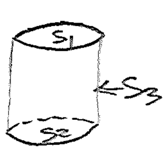
▸ Immediately follows from the divergence thm and the equality $\operatorname{div}(\operatorname{curl}(\vec{F}))=0$ ∎

Ex4: Verify the divergence theorem for $\vec{F} = \langle x^3, y^3, z^2\rangle$ and E – cylindrical volume $x^2+y^2\le 9$ with $0\le z\le 2$.

▸ $\operatorname{div}(\vec{F}) = 3x^2+3y^2+2z$. $\Rightarrow \iiint_B \operatorname{div}(\vec{F})\, dV = \int_0^{2\pi}\int_0^3\int_0^2 (3r^2+2z)\cdot r\, dz\, dr\, d\theta =$

$$= \int_0^{2\pi}\int_0^3 (6r^3+4r)\, dr\, d\theta = 2\pi\cdot\left(\frac{6}{4}r^4+2r^2\right)\Big|_{r=0}^{r=3} = 2\pi\cdot\left(\frac{3\cdot 81}{2}+18\right) = \boxed{279\pi}$$

For the surface integral, we split S into 3 parts



S_1: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2\rangle \Rightarrow \vec{r}_r\times\vec{r}_\theta = \langle 0,0,r\rangle$ – looks outside the cylinder

$$\Rightarrow \iint_{S_1} \vec{F} dS = \int_0^{2\pi}\int_0^3 4r\, dr\, d\theta = 36\pi.$$

S_2: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle \Rightarrow \vec{r}_r\times\vec{r}_\theta = \langle 0,0,r\rangle \Rightarrow$ need take opposite $\langle 0,0,-r\rangle$

$$\iint_{S_2} \vec{F} dS = \int_0^{2\pi}\int_0^3 0\, dr\, d\theta = 0.$$

S_3: $\vec{r}(\theta,z) = \langle 3\cos\theta, 3\sin\theta, z\rangle \Rightarrow \vec{r}_\theta\times\vec{r}_z = \langle 3\cos\theta, 3\sin\theta, 0\rangle$ – points outwards.

$\iint_{S_3} \vec{F} dS = \int_0^{2\pi}\int_0^2 (81\cos^4\theta + 81\sin^4\theta)\, dz\, d\theta = 243\pi$. And so $\iint_S \vec{F} dS = 36\pi + 243\pi = \boxed{279\pi}$ ∎

Ex 5: Evaluate $\iint_S \vec{F}\,d\vec{S}$, where $\vec{F} = \langle xy, -\frac{1}{2}y^2, z\rangle$ and the surface S consists of 3 surfaces: $z=4-3x^2-3y^2$, $1\le z\le 4$. on the top, $x^2+y^2=1$, $0\le z\le 1$ on the sides, and $z=0$ on the bottom.

$\operatorname{div}(\vec{F}) = y - y + 1 = 1.$

In cylindrical coordinates the solid E bounded by S is given by

$0\le \theta\le 2\pi, \quad 0\le r\le 1, \quad 0\le z\le 4-3r^2.$

Thus: $$\iint_S \vec{F}\,d\vec{S} = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (4r-3r^3)\,dr\,d\theta =$$
$$= \int_0^{2\pi}\left(2r^2-\frac{3}{4}r^4\right)\Big|_{r=0}^{r=1} d\theta = \boxed{\frac{5}{2}\pi}$$

Ex6: Evaluate the flux of $\vec{F} = \langle xy^2, yz^2, x^2z+e^x\rangle$ across the sphere S of radius 3 centered at the origin with an "inside" orientation.

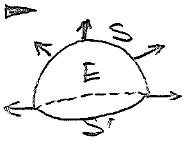
$\operatorname{div}\vec{F} = y^2+z^2+x^2$

Let B be the ball of radius 3 bounded by S.

Then, by divergence thm (note a wrong orientation leads to "$-$"):

$$\iint_S \vec{F}\,d\vec{S} = -\iiint_B (x^2+y^2+z^2)\,dV = -\int_0^{\pi}\int_0^{2\pi}\int_0^3 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi =$$
$$= -\int_0^3 \rho^4 d\rho\cdot\int_0^{\pi}\sin\phi\,d\phi\cdot\int_0^{2\pi}d\theta = -\frac{3^5}{5}\cdot 2\cdot 2\pi = \boxed{-\frac{243\cdot 4\pi}{5}}$$

Ex 7: Evaluate the integral $\iint_S \vec{F}\,d\vec{S}$, where $\vec{F} = \langle x+e^z, y+\sin(z^2), 2z+1\rangle$ and S is the hemisphere $\begin{cases} x^2+y^2+z^2=1 \\ z\ge 0\end{cases}$ oriented upwards.



To apply the divergence theorem, we need to "close" the surface S. There are many ways to do this, but the simplest is to add disk S' on the bottom, oriented "downwards".

Then: $$\iint_S \vec{F}\,d\vec{S} + \iint_{S'} \vec{F}\,d\vec{S} = \iint_{S\cup S'} \vec{F}\,d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E 4\,dV = 4\,\mathrm{Vol}(E) = 4\cdot\frac{4\pi}{3}\cdot\frac{1}{2} = \frac{8\pi}{3}$$

We compute $\iint_{S'} \vec{F}\,d\vec{S}$ directly: $\iint_{S'} \vec{F}\,d\vec{S} = \iint_{S'} \vec{F}\cdot(-\vec{k})\,dA = \iint_{S'}(-1)\,dA = -\mathrm{Area}(S') = -\pi.$

So: $\iint_S \vec{F}\,d\vec{S} = \frac{8\pi}{3} - (-\pi) = \frac{11\pi}{3}$