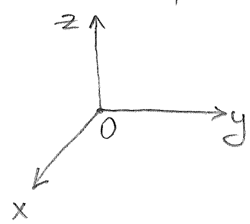
Lecture #1

08/30/2018

- Help #1: Office hours, to be held on Tu/Wed, 2-3pm, LOM 219-C
  Help #2: Peer tutors (additional help with the material).
- Plan for the first two weeks: Sections 12.1 – 12.5 of your textbook.

Recall that any point on a line is represented by a number (its coordinate). Likewise any point in the plane or the space can be represented by an ordered pair (a,b) or triple (a,b,c) of numbers, respectively.

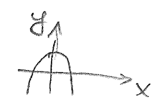


For the latter, we need to pick the origin O and choose three orthogonal axes: x, y, z (as on the pic.)
In this way the point P with coordinates (a,b,c) satisfies the following properties:

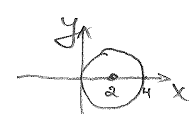
* a is the directed distance from P to the yz-plane
* b is the directed distance from P to the xz-plane
* c is the directed distance from P to the xy-plane

In this class, we will mostly study 3D and 2D objects.
Recall that in 2D (2-dimensional) geometry, a graph of an eq-n involving x, y is a curve in $\mathbb{R}^2$.

Ex1: (a) Draw a graph of $y = 1 - x^2$



(b) Draw a graph of $x^2 - 4x + y^2 = 0$



Likewise, in 3D, an equation in x, y, z represents a surface in $\mathbb{R}^3$.

Ex 2: (a) What surface in $\mathbb{R}^3$ is represented by the equation $x = 2$?
(Plane parallel to yz-plane and passing through $(2,0,0)$)

(b) What surface in $\mathbb{R}^3$ is represented by the equation $x^2 + y^2 = 4$?
(Circular cylinder with radius 2 and whose axis is z-axis).

(c) What surface in $\mathbb{R}^3$ is represented by $y = z$?
(Plane containing x-axis and intersecting yz-plane at the line $y = z$).

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Recall that a distance b/w two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ in the plane is given by $\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$, due to Pythagorean thm.
Likewise, the distance between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ in the space equals

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

Ex 3: (a) Find an equation of a sphere with radius 3 and center $(2,1,3)$

$[(x-2)^2+(y-1)^2+(z-3)^2=3^2 \iff x^2+y^2+z^2-4x-2y-6z=5]$

(b) Show that $x^2+y^2+z^2+2x-2y-4z-10=0$ is the eqn of a sphere. Find its radius & center

[Rewrite as $(x+1)^2+(y-1)^2+(z-2)^2=16 \Rightarrow$ center $=(-1,1,2)$, radius $=4$].

(c) Find an equation of a sphere passing through $(4,5,3)$ and whose center is $(1,2,1)$.

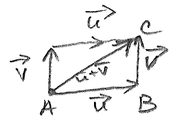
$\left[(x-1)^2+(y-2)^2+(z-1)^2=\underbrace{(4-1)^2+(5-2)^2+(3-1)^2}_{22} \iff x^2+y^2+z^2-2x-4y-2z=16\right]$

Recall that a vector is a quantity that has both magnitude and direction. We represent a vector by a directed arrow, the length of which equals the magnitude.

One of the most common examples is the displacement vector $\overrightarrow{AB}$, where A denotes the initial position of an object, while B - its terminal position.
Note that $\vec{v}$, $\vec{u}$ do not coincide, but are equal vectors, since they look in the same direction and have the same magnitude.

There are two operations one can do with vectors:

(1) add vectors $\vec{u}, \vec{v} \rightsquigarrow \vec{u}+\vec{v}$



"Parallelogram Law"

in terms of the above example, if an object moves from A to B in step 1 and then from B to C in step 2, then its total displacement $\overrightarrow{AC}$ is a sum of the two displacement vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ in each step.

(2) Multiplying a vector $\vec{v}$ by a scalar c.

Note that if $c=0$, then $0\cdot\vec{v}$ is ZERO, while if $c<0$, then $c\cdot\vec{v}$ looks in the opposite direction

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One of the most convenient ways to treat vectors is by introducing a coordinate system first. Then, if we place the initial point of a vector $\vec{v}$ at the origin, then the terminal point of $\vec{v}$ has coordinates (v_1, v_2, v_3) (or (v_1, v_2) if we are in the plane).

Notation: These coordinates are called the components of $\vec{v}$ and we write
$$\vec{v} = \langle v_1, v_2, v_3 \rangle \quad (\text{or } \vec{v} = \langle v_1, v_2 \rangle \text{ if in 2D}).$$

In particular, given two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the space, the displacement vector $\vec{AB}$ equals
$$\boxed{\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle}$$

Ex 4: (a) Find the vector $\vec{v}$ represented by the directed line segment with initial point $A(-1,3,2)$ and terminal point $B(3,2,0)$.
[$\vec{v} = \langle 3-(-1), 2-3, 0-2 \rangle = \langle 4, -1, -2 \rangle$]

(b) Find the length/magnitude of $\vec{v}$ from part (a).
[$|\vec{v}| = \sqrt{4^2 + (-1)^2 + (-2)^2} = \sqrt{21}$].

The convenience of this approach is that all operations on vectors are done component-wise, i.e.

(1) If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

(2) If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, then $c \cdot \vec{a} = \langle ca_1, ca_2, ca_3 \rangle$.

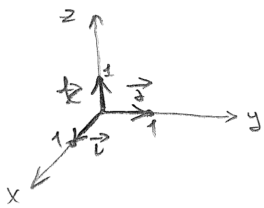
Ex 5: (a) If $\vec{v} = \langle 1, 2 \rangle$, $\vec{w} = \langle 3, 1 \rangle$, sketch the vectors $\vec{v} + \vec{w}$, $\vec{v} - 2\vec{w}$

(b) If $\vec{v} = \langle 2, -1, 1 \rangle$, $\vec{w} = \langle -1, 0, 2 \rangle$, find $\vec{v} - 3\vec{w}$ and its magnitude.

(c) Find the vector $\vec{v}$ whose magnitude is twice that of $\vec{u} = \langle 1, 3, 5 \rangle$, while the direction is opposite to that of $\vec{u}$.
[$\vec{v} = -2 \cdot \langle 1, 3, 5 \rangle = \langle -2, -6, -10 \rangle$].

There are 3 distinguished vectors (given a coordinate system):

$$\boxed{\vec{i} = \langle 1,0,0\rangle, \quad \vec{j} = \langle 0,1,0\rangle, \quad \vec{k} = \langle 0,0,1\rangle}$$ ← standard basis vectors



These are the length 1 vectors pointing in the direction of positive x-, y-, z-axes, respectively.

Then: $\boxed{\vec{v} = \langle v_1, v_2, v_3\rangle = v_1 \cdot \vec{i} + v_2 \cdot \vec{j} + v_3 \cdot \vec{k}}$ in 3D

$\boxed{\vec{v} = \langle v_1, v_2\rangle = v_1 \cdot \vec{i} + v_2 \cdot \vec{j}}$ in 2D

Ex 6: If $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - 2\vec{k}$, find $\vec{a} + 2\vec{b}$ and $|\vec{a} + 2\vec{b}|$.

Recall that a vector $\vec{v}$ is called a unit vector if $|\vec{v}| = 1$.

Ex 7: Find a unit vector in the direction opposite to $\langle -3,4\rangle$. Express via $\vec{i}, \vec{j}$.

$\left[\ \vec{v} = -\frac{1}{\sqrt{3^2+4^2}}\langle -3,4\rangle = \langle \frac{3}{5}, -\frac{4}{5}\rangle = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\ \right]$.

So far we have discussed addition of vectors and multiplication by a scalar.

Question: Is there a notion of product for vectors?

Today we will discuss the so-called dot product, while next time we will learn a more complicated notion of cross product (only in 3D).

Def: If $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, then their dot product (a.k.a. inner product) is the number $\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + a_3 b_3$

Note: • The result is a number, not a vector!

• Also if $\vec{a} = \langle a_1, a_2\rangle$, $\vec{b} = \langle b_1, b_2\rangle$ are 2D vectors, then we similarly define $\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2$.

THEOREM: If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta} \Leftrightarrow \boxed{\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}}$$

Rmk: This easily follows from the Law of Cosines – see p.808 of your textbook.

Ex 8: (a) Verify that $3\vec{i}-\vec{j}+2\vec{k}$ is perpendicular to $-2\vec{i}+4\vec{j}+5\vec{k}$.

[Compute their dot product $\Rightarrow$ get ZERO $\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$].

(b) Are the vectors $\langle 1,-2,1\rangle$ and $\langle 2,3,1\rangle$ orthogonal? If not, determine if the angle between these vectors is obtuse or acute? Determine the angle explicitly.

$[\theta = \cos^{-1}\left(-\frac{3}{\sqrt{6}\cdot\sqrt{14}}\right) = \cos^{-1}\left(-\frac{3}{\sqrt{84}}\right)]$

(c) Use vectors to decide if the triangle with vertices $P(1,2)$, $Q(2,1)$, $R(3,4)$ is right-angled.

(d) Find the value of x, such that the vector $\langle x,-1,2x\rangle$ is orthogonal to $\langle 1,3,-1\rangle$.

(e) Find the acute angle between the lines $x+y=1$ and $x-2y=-2$