Lecture #2

09/04/2018

* Organization

- Office hours: Tue & Wed, $2^{00}-3^{00}$ pm, LOM 219-C.
- The 1st homework is due this Thursday, Sept. 6.
  ↳ Please try write it down today, so that if you got any questions you may come tomorrow to the office hours.
- The homework should be submitted on paper - please staple pages & write your name.

* Reminder of the material from the previous lecture

- Ask if there are any questions from last lecture.
- Last time we discussed:
  (1) computing the distance between 2 points in $\mathbb{R}^2$ or $\mathbb{R}^3$.
  (2) equation of a sphere, completing squares to find center & radius.
  (3) vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$; addition and multiplication by scalars
  (4) components of vectors
  (5) dot product: algebraic f-la/definition + geometric meaning.

Let us warm up by doing a couple of examples relevant to Lecture 1.

Ex1: Find the lengths of the sides of the triangle PQR with $P(2,2,2)$, $Q(4,1,1)$, $R(1,1,1)$. Is it a right and/or isosceles triangle?

► $PQ = \sqrt{2^2+1^2+1^2} = \sqrt{6}$, $QR = \sqrt{3^2+0^2+0^2} = 3$, $PR = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.
Clearly not isosceles, but it is a right Δ as $QR^2 = PQ^2 + PR^2$ ∎

Ex2: Determine the angle between vectors $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$ and $\vec{b} = 3\vec{i} + \vec{j}$.

► First, let us note that component-wise $\vec{a} = \langle 1, -2, 1 \rangle$, $\vec{b} = \langle 3, 1, 0 \rangle$ while their magnitudes are $|\vec{a}| = \sqrt{6}$, $|\vec{b}| = \sqrt{10}$. Next, use dot product
$|\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = \vec{a} \cdot \vec{b} = 1 \cdot 3 + (-2) \cdot 1 + 1 \cdot 0 = 1 \Rightarrow \cos\theta = \frac{1}{\sqrt{60}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{60}}\right)$ ∎

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Recall: If θ denotes the angle between nonzero vectors $\vec{a}$ and $\vec{b}$, then:

- If $\vec{a}\cdot\vec{b}=0 \Rightarrow \theta=90°$
- If $\vec{a}\cdot\vec{b}>0 \Rightarrow \theta<90°$
- If $\vec{a}\cdot\vec{b}<0 \Rightarrow \theta>90°$.

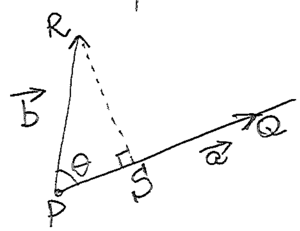
Ex3: Find the value of x, such that the vector $\langle x,-1,2x\rangle$ is orthogonal to $\langle 1,3,-1\rangle$.

$$0=\langle x,-1,2x\rangle\cdot\langle 1,3,-1\rangle = x-3-2x=-x-3 \iff \underline{\underline{x=-3}}$$

∎

* Projections

Given two vectors $\vec{a}$ and $\vec{b}$, consider their representations with the same initial point $\vec{PQ}$ and $\vec{PR}$, respectively.



Let S be the foot of the perpendicular from the point R to the line containing $\vec{PQ}$.

Definition: (1) The vector $\vec{PS}$ is called the vector projection of $\vec{b}$ onto $\vec{a}$, and it is denoted $\text{proj}_{\vec{a}}\vec{b}$.

(2) The scalar projection of $\vec{b}$ onto $\vec{a}$, denoted $\text{comp}_{\vec{a}}\vec{b}$, is the signed magnitude of the vector projection.

- positive if $\vec{PS}$ & $\vec{PQ}$ look the same way, equiv if $\theta<90°$
- negative if $\vec{PS}$ & $\vec{PQ}$ look the opposite way, equiv. if $\theta>90°$.
- zero if $\vec{a}\perp\vec{b}$, equiv. $\theta=90°$.

From the picture we immediately see that

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cdot\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

How about $\text{proj}_{\vec{a}}\vec{b}$? Any suggestions on how to compute it?

Hint: Vector $\vec{PS}$ is parallel to $\vec{a}=\vec{PQ}$ and we know its magnitude.

Answer: To get $\overrightarrow{PS}$ we need to multiply $\text{comp}_{\vec a}\vec b$ by the unit vector in the direction of $\vec a$, which has the form $\frac{\vec a}{|\vec a|}$.

So: $$\boxed{\text{proj}_{\vec a}\vec b = \frac{\vec a\cdot\vec b}{|\vec a|^2}\cdot\vec a}$$

Ex 4: Find the scalar and vector projections of $\vec b$ onto $\vec a$.

(a) $\vec a = \langle -3,4\rangle$, $\vec b = \langle 2,1\rangle$

$\text{comp}_{\vec a}\vec b = \frac{\vec a\cdot\vec b}{|\vec a|} = \frac{-2}{5}$

$\text{proj}_{\vec a}\vec b = -\frac{2}{5}\cdot\frac{\langle -3,4\rangle}{5} = \left\langle \frac{6}{25}, -\frac{8}{25}\right\rangle$ ∎

(b) $\vec a = 2\vec i + \vec j - 2\vec k$, $\vec b = \vec i + 2\vec j + 3\vec k$

$\text{comp}_{\vec a}\vec b = \frac{\vec a\cdot\vec b}{|\vec a|} = -\frac{2}{3}$

$\text{proj}_{\vec a}\vec b = -\frac{2}{3}\cdot\frac{\langle 2,1,-2\rangle}{3} = \left\langle -\frac{4}{9}, -\frac{2}{9}, \frac{4}{9}\right\rangle$ ∎

Ex 5*: Suppose that $\vec a$ and $\vec b$ are nonzero vectors.

(a) Under which circumstances is $\text{comp}_{\vec a}\vec b = \text{comp}_{\vec b}\vec a$?

(b) Under which circumstances is $\text{proj}_{\vec a}\vec b = \text{proj}_{\vec b}\vec a$?

Remark: One of the natural applications of projections is calculating work. If the constant force $\vec F = \overrightarrow{PR}$ moves the object from P to Q, so that $\vec D = \overrightarrow{PQ}$ is the displacement vector, then work done by this force is defined as the product of the component of the force along $\vec D$ and the distance moved:

$$\boxed{W = (|\vec F|\cos\theta)\cdot|\vec D| = \vec F\cdot\vec D}$$

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* Cross product

Last time we defined the dot product of two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, which is a number. However, for 2 vectors in $\mathbb{R}^3$ there is another operation which produces another vector in $\mathbb{R}^3$.

Note: this does not work for vectors in $\mathbb{R}^2$.

Definition: Given two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, define their cross product as follows

$$\boxed{\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle}$$

An easy way to remember this formula is to write down the matrix
$\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$ and compute the following expression

$$\vec{i} \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad (\text{"determinant"})$$

where the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{\text{def}}{:=} ad - bc$.

Ex6: (a) Find the cross product $\vec{a} \times \vec{b}$ of $\vec{a} = \langle 2, 1, 3 \rangle$, $\vec{b} = \langle 1, 2, -1 \rangle$

(b) Verify that it is perpendicular to both $\vec{a}$ and $\vec{b}$.

(c) Find the cross product $\vec{b} \times \vec{a}$ for the above vectors

(d) Find the cross product $\vec{a} \times \vec{a}$.

(a) $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = -7\vec{i} + 5\vec{j} + 3\vec{k} = \langle -7, 5, 3 \rangle$

(b) $\langle -7, 5, 3 \rangle \cdot \langle 2, 1, 3 \rangle = 0$, $\langle -7, 5, 3 \rangle \cdot \langle 1, 2, -1 \rangle = 0$

(c) $\vec{b} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \langle 7, -5, -3 \rangle$

(d) $\vec{a} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{vmatrix} = \langle 0, 0, 0 \rangle = \vec{0}$

As illustrated by the previous problem, cross product always satisfies:

(a) $\vec{a} \times \vec{a} = \vec{0}$

(b) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ ← discuss why

(c) $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Moreover, the direction of $\vec{a} \times \vec{b}$ is given by the "right-hand rule", i.e. if the fingers of your right hand curl in the direction of a rotation (through an angle $< 180°$) from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{a} \times \vec{b}$

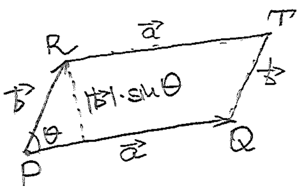
Question: What about the magnitude of $\vec{a} \times \vec{b}$?

Theorem (see p.817 of textbook): If θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, then $\boxed{|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\theta}$

Corollary: $\vec{a} \times \vec{b} = 0$ iff $\vec{a}$ is parallel to $\vec{b}$.

Upshot: $\vec{a} \times \vec{b}$ is the vector perpendicular to both $\vec{a}, \vec{b}$, whose orientation is determined by the right-hand rule, and whose length is $|\vec{a}| \cdot |\vec{b}| \cdot \sin\theta$.

Another geometric interpretation of the quantity $|\vec{a}||\vec{b}|\sin\theta$ becomes clear from the picture below:



The area of the parallelogram with sides $\vec{a}$ and $\vec{b}$ equals exactly $|\vec{a}| \cdot (|\vec{b}| \cdot \sin\theta) = |\vec{a} \times \vec{b}|$.

So: The length of the cross product $\vec{a} \times \vec{b}$ equals the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

$\Downarrow$

The area of the triangle PQR equals $\frac{1}{2}|\vec{a} \times \vec{b}|$.

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Ex7: Find a vector perpendicular to the plane containing 3 points $P(1,2,3)$, $Q(2,1,1)$, $R(3,0,1)$

E.g. $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1,-1,-2 \rangle \times \langle 2,-2,-2 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -2 \\ 2 & -2 & -2 \end{vmatrix} = \langle -2,-2,0 \rangle$

Or you may write e.g. $\langle 1,1,0 \rangle$.

Ex8: Find the area of the triangle PQR with P, Q, R - as in Ex7.

$\text{Area}(PQR) = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{4+4} = \sqrt{2}$

Definition: The scalar triple product of three vectors $\vec{a}, \vec{b}, \vec{c}$ is defined as $\vec{a} \cdot (\vec{b} \times \vec{c})$ (which is a number!).

Geometric meaning: The absolute value $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ equals the volume of the parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$.

Ex 9: Verify that the vectors $\vec{a} = \langle 1,-1,-2 \rangle$, $\vec{b} = \langle 2,-2,-2 \rangle$, $\vec{c} = \langle -1,1,5 \rangle$ are coplanar, i.e. lie in the same plane.

Ex 10: Verify $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ for any three vectors in $\mathbb{R}^3$.