Lecture #3 09/06/2018

* Organization

- 1^{st} hwk shall be submitted by the end of today's class
- 2^{nd} hwk will be posted today
- Office Hours: Tue, Wed 2-3pm, LOM 219-C.

* Reminder of the material from the previous lecture

- scalar and vector projection
- cross product: explicit formula + geometric meaning of $|\vec{a}\times\vec{b}|$.

! Important: dot product of two vectors is a number
cross product of two vectors in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$

Ex1: Suppose $\vec{a}$ and $\vec{b}$ are nonzero vectors.
(a) Under which circumstances is $\text{comp}_{\vec{a}}\vec{b} = \text{comp}_{\vec{b}}\vec{a}$?
(b) Under which circumstances is $\text{proj}_{\vec{a}}\vec{b} = \text{proj}_{\vec{b}}\vec{a}$?

▶ (a) $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ $\qquad$ $\text{comp}_{\vec{b}}\vec{a} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|}$

So: $\text{comp}_{\vec{a}}\vec{b} = \text{comp}_{\vec{b}}\vec{a} \iff \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|}$
$\vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b}$

$\Rightarrow$ $\vec{a}\cdot\vec{b} = 0$ or $|\vec{a}| = |\vec{b}|$.

Answer: $\vec{a}$ & $\vec{b}$ are orthogonal or of the same length

(b) $\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\underbrace{\dfrac{\vec{a}}{|\vec{a}|}}_{\text{unit vector in direction of } \vec{a}}$ $\qquad$ $\text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|}\cdot\underbrace{\dfrac{\vec{b}}{|\vec{b}|}}_{\text{unit vector in direction of } \vec{b}}$.

First of all comparing magnitudes and using (a): $\vec{a}\perp\vec{b}$ or $|\vec{a}| = |\vec{b}|$.

- If $\vec{a}$ is orthogonal to $\vec{b}$, then both vector projections are ZERO
- If $\vec{a}$ is not orthogonal to $\vec{b}$, but $|\vec{a}| = |\vec{b}|$, then we see from above that $\vec{a} = \vec{b}$.

Answer: $\vec{a}$ & $\vec{b}$ are orthogonal or equal ∎

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Ex2: Verify that $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ for any three vectors in $\mathbb{R}^3$.

▶ If $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, $\vec{c} = \langle c_1, c_2, c_3\rangle$, then

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \langle a_1, a_2, a_3\rangle \cdot \langle b_2c_3 - b_3c_2,\ b_3c_1 - b_1c_3,\ b_1c_2 - b_2c_1\rangle$$
$$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

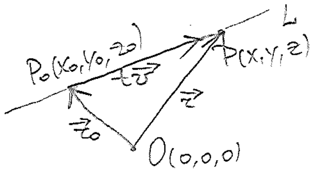
$$(\vec{a}\times\vec{b})\cdot\vec{c} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle \cdot \langle c_1, c_2, c_3\rangle$$
$$= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3$$

Now we see these two expressions coincide! ◻

Note: The common value of $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ is called the scalar triple product and the absolute value of it is equal to the volume of the parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$.

* Equation of lines

A line L in $\mathbb{R}^3$ is determined once we know a point $P_0(x_0, y_0, z_0)$ on L as well as the direction of L, which is described by a vector $\underset{\langle a,b,c\rangle}{\vec{v}}$ parallel to L.



$$\vec{OP} = \underbrace{\vec{OP_0}}_{\langle x_0, y_0, z_0\rangle} + \underbrace{\vec{P_0P}}_{t\cdot\langle a,b,c\rangle = \langle ta, tb, tc\rangle}$$
$$\Downarrow$$
$$P(x_0+ta,\ y_0+tb,\ z_0+tc)$$

Upshot: (1) A vector equation of the line L is: $\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$ (t: changing parameter)

(2) Once we fix a coordinate system, we get a parametric equation of the line L:

$$\boxed{x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct}$$

! Important: Any line has a lot of different parametric equations due to (1) change of a "starting point" P_0 on L
(2) change of the vector $\vec{v}$ (any multiple of it works)

E.g.: $x = t, y = 2t, z = 3t+1$ and $x = 1+3t, y = 2+6t, z = 4+9t$ determine the same line

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Ex3: (a) Find a vector equation and parametric eq-n for the line passing through $(2,4,1)$ and parallel to the vector $\langle 1,-1,\tfrac{1}{2}\rangle$.

(b) Find a point on this line whose x-coordinate is ZERO.

$\blacktriangleright$ $(0,6,0)$ ■

Def: The numbers a,b,c are called direction numbers of the line L.

Eliminating t, we obtain the symmetric eq-n of the line L (assuming $a,b,c\neq 0$):

$$\boxed{\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}}$$

Ex4: (a) Find parametric and symmetric equations of the line that passes through $A(2,1,3)$ and $B(3,4,5)$.

(b) Find an equation of the line segment between A & B in part (a).

$\blacktriangleright$ $\vec{r}=\langle 2,1,3\rangle + t\underbrace{\langle 1,3,2\rangle}_{\vec{AB}}, \; 0\le t\le 1$ ■

In general, the line segment between points A and B (with $\vec{OA}=\vec{r_0}$, $\vec{OB}=\vec{r_1}$) is given by $\boxed{\vec{r}(t)=\vec{r_0}+t(\vec{r_1}-\vec{r_0}),\; 0\le t\le 1}$

Ex5: Determine whether the lines L_1, L_2 are parallel, skew or intersecting:

L_1: $x=-2+t,\; y=1-2t,\; z=3t$

L_2: $x=1-s,\; y=2+3s,\; z=-2s$

! Important: When working with two lines you should use different parameters, e.g. t and s, rather than the same t!

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* Equations of planes

A plane in $\mathbb{R}^3$ is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane.
normal vector

Then a point $P(x,y,z)$ belongs to this plane iff $\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$.
$\overrightarrow{OP} - \overrightarrow{OP_0} = \vec{r} - \vec{r_0}$

So: A vector eq-n of the plane is

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \text{or equiv.} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}}$$

Once we fix a coordinate system and set $\vec{n} = \langle a, b, c \rangle$, we get a scalar eq-n of the plane through point $P_0(x_0, y_0, z_0)$ and $\perp \vec{n} = \langle a, b, c \rangle$:

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Finally setting $d := -ax_0 - by_0 - cz_0$, we obtain a linear eq-n of the plane:

$$\boxed{ax + by + cz + d = 0}$$

Ex6: Find an equation of the plane through the point $(3, -1, 4)$ and perpendicular to the line $L: x = 2 - t, y = 3 + 2t, z = 1 - 7t$.

Ex7: Find an equation of the plane that passes through the points $P(1,2,3)$, $Q(2,1,1)$, $R(3,0,1)$

Hint: Can choose normal vector to be $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -2, -2, 0 \rangle$ - see Lecture 2.

Ex8: Find the point at which the line $L: x = -1 - t, y = 2 + 3t, z = -10t$ intersects the plane $2x + y - \frac{z}{5} - 1 = 0$

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Given two planes, they are either parallel (may coincide) or intersect in a line.

(1) The direction of this line may be determined e.g. by computing $\vec{n_1} \times \vec{n_2}$, where $\vec{n_1}, \vec{n_2}$ - normal vectors to the planes.

(2) The angle θ between the planes is determined as the acute angle between their normal vectors.

Ex 9: (a) Find an angle between the planes $x-y+z=2$ and $2x+y-2z=1$

(b) Find a parametric eq-n for the line of their intersection.

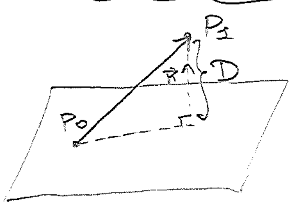
(a) $\vec{n_1} = \langle 1,-1,1 \rangle$, $\vec{n_2} = \langle 2,1,-2 \rangle \Rightarrow \cos\theta = \left| \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|} \right| = \frac{1}{3\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right)$

(b) $\vec{n_1} \times \vec{n_2} = \langle 1,4,3 \rangle$

e.g. point $(1,-1,0)$ - on the line $\Big\} \rightarrow$ $\underline{x=1+t,\ y=-1+4t,\ z=3t}$

line of the intersection. ∎

* Distance from a point to a plane



Given a point $P_1(x_1,y_1,z_1)$ and a plane passing through $P_0(x_0,y_0,z_0)$ and with normal vector $\vec{n}$, $\langle a,b,c \rangle$ the distance from P_1 to this plane equals

$$\boxed{D = \frac{|\vec{P_0P_1} \cdot \vec{n}|}{|\vec{n}|} = \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}}$$

Hence, if the plane is given by $ax+by+cz+d=0$, then

$$\boxed{D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

Ex 10: Compute a distance from $P(1,2,-3)$ to the plane $x-2y+3z=8$.

$D = \frac{20}{\sqrt{14}}$ ∎