Lecture #4 09/11/2018

* Organizational
  - Office hours Tu, Wed $2^{00} - 3^{00}$ pm, LOM 219-C
  - Peer tutors : additional help.

* Reminder of the material from the previous lecture

Last time we discussed equations of lines and planes

- Lines : (1) $\vec{r} = \vec{r_0} + t \cdot \vec{v}$ ← "vector equation" (t varies over all real numbers)

  (2) $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$ ← "parametric equation"

  (here a, b, c – coordinates of $\vec{v}$)

  (3) $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ ← "linear equation"

- Planes : (1) $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$ ← "vector equation" ($\vec{n}$ is orthogonal to the plane)

  (2) $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ ← "scalar equation"

  (3) $ax + by + cz + d = 0$ ← "linear equation" ($d := -ax_0 - by_0 - cz_0$)

Math ingredients from last time:

1) If you need to construct a line passing through 2 points A, B, you may pick P_0 to be either A or B, while $\vec{v}$ to be either $\vec{AB}$ or $\vec{BA}$.

2) If you need to construct a plane containing 3 points A, B, C (which do not lie on 1 line), you may pick P_0 as one of A, B, C, and $\vec{n}$ e.g. as $\vec{AB} \times \vec{AC}$.

3) Two lines in $\mathbb{R}^3$ are either parallel, intersect at 1 pt, or skew.
   To check if they are parallel – look at direction vectors
   To check if non-parallel lines intersect – solve a system of 3 equations on 2 variables.

4) Given a line and a plane, one of 3 cases occurs: line belongs to the plane, line is parallel to the plane and doesn't belong to it, line intersects plane at exactly 1 pt.
   To check which holds : plug eq-n of a line into an equation of a plane.

5) Two planes → either parallel ← if normal vectors are parallel
   → or intersect at a line

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Example from the end of last lecture:

Ex1: (a) Find an angle between the planes $x-y+z=2$, $2x+y-2z=1$

(b) Find a parametric equation for the line of their intersection

▸ (a) $\vec{n_1}=\langle 1,-1,1\rangle$, $\vec{n_2}=\langle 2,1,-2\rangle \Rightarrow \cos\theta = \left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}\right| = \frac{1}{3\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right)$

(b) First, the line of intersection is parallel to

$$\vec{v}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 1\\ 2 & 1 & -2\end{vmatrix}=\langle 1,4,3\rangle.$$

To find a point on the line, let's try to solve $\begin{cases}x-y+z=2\\ 2x+y-2z=1\end{cases}$ by setting $z=0$.

Then $\left.\begin{cases}x-y=2 \Rightarrow x=2+y\\ 2x+y=1\end{cases}\right\} \Rightarrow 2(2+y)+y=1 \Rightarrow y=-1 \Rightarrow x=1$

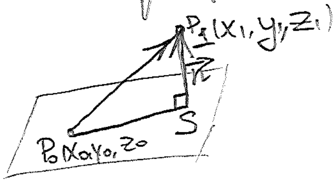
So: $(1,-1,0)$ – on the line.

Thus, line of the intersection is given by $x=1+t,\ y=-1+4t,\ z=3t$ ▣

• Last time, we learnt the distance formula from the point $P_1(x_1,y_1,z_1)$ to the plane $ax+by+cz+d=0$:

$$\boxed{D=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

Let's explain where it is coming from. Let P_0 be any point on the plane.



$$D=|\text{comp}_{\vec{n}}\vec{b}|=\frac{\vec{n}\cdot\vec{b}}{|\vec{n}|}=\left|\frac{\langle a,b,c\rangle\cdot\langle x_1-x_0,\ y_1-y_0,\ z_1-z_0\rangle}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\frac{|ax_1+by_1+cz_1-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

Example from the end of last time

Ex2: Compute a distance from $P(1,2,-3)$ to the plane $x-2y+3z=8$

▸ Use above f-la (note that $d=-8$, not 8!):

$$D=\frac{|1\cdot 1+(-2)\cdot 2+3\cdot(-3)-8|}{\sqrt{1^2+2^2+3^2}}=\frac{20}{\sqrt{14}}$$ ▣

* Today: vector functions (13.1–13.4 in the textbook).

A vector function is a function whose domain is a subset of $\mathbb{R}$ and whose range is a set of vectors

In this class, we will be mostly interested in the case when values are el-s of $\mathbb{R}^3$.
Thus, a vector function is an assignment

$$t \longmapsto \vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \in \mathbb{R}^3.$$

component functions

Domain: the set of t, where $\vec{r}(t)$ is defined
Range: all possible values of $\vec{r}(t)$ as t varies in domain.

* Limits and Continuity

The limit of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is defined component-wise

$$\boxed{\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle}$$ if all 3 limits exist and doesn't exist otherwise

A vector function $\vec{r}(t)$ is continuous at $a \in \mathbb{R}$ if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

Note: $\vec{r}$ is continuous at a if all 3 components are continuous at a.

Ex3: Find $\lim_{t\to 0} \vec{r}(t)$, where $\vec{r}(t) = \left\langle t + \frac{1}{t+2}, \frac{e^t - 1}{2t}, \frac{\sin 5t}{2t} \right\rangle$

$$\left.\begin{aligned}
&\lim_{t\to 0}\left(t + \frac{1}{t+2}\right) = 0 + \frac{1}{2} = \frac{1}{2} \\
&\lim_{t\to 0} \frac{e^t - 1}{2t} \overset{\text{L'Hospital}}{=} \lim_{t\to 0} \frac{(e^t-1)'}{(2t)'} = \lim_{t\to 0} \frac{e^t}{2} = \frac{e^0}{2} = \frac{1}{2} \\
&\lim_{t\to 0} \frac{\sin(5t)}{2t} \overset{\text{L'Hospital}}{=} \lim_{t\to 0} \frac{\cos(5t)\cdot 5}{2} = \frac{5}{2}\cos(0) = \frac{5}{2}
\end{aligned}\right\} \Rightarrow \lim_{t\to 0} \vec{r}(t) = \left\langle \frac{1}{2}, \frac{1}{2}, \frac{5}{2} \right\rangle$$

* Space Curves

Let f, g, h be continuous $\mathbb{R}$-valued fun-s on an interval I. Then the locus C of all points $(x,y,z) \in \mathbb{R}^3$ s.t. $x = f(t)$, $y = g(t)$, $z = h(t)$, $t \in I$ is called a space curve

Here, t – parameter, while the above eq-s are parametric eq-s of C

Note: Any continuous vector f-n $\vec{r}$ defines a space curve C, traced out by the tip of $\vec{r}(t)$

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Ex4: Describe the curve defined by the vector functions:

(a) $\vec{r}(t) = \langle 1+2t, 3-5t, 7+t \rangle$

- line passing through $(1,3,7)$ and parallel to $\langle 2,-5,1 \rangle$ ▫

(b) $\vec{r}(t) = \langle 2\cos t, 2\sin t, 1 \rangle$

- circle of radius 2, centered at $(0,0,1)$, parallel to xy-plane ▫

(c) $\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$

- helix

lies on the cylinder

whose projection to xy-plane is a circle, centered at $(0,0)$ of radius 2 ▫

Ex5: Find a vector function that represents

(a) a line segment between $P(1,2,3)$ and $Q(7,5,6)$.

- $\vec{r}(t) = \langle 1,2,3 \rangle + t\langle 7-1, 5-2, 6-3 \rangle = \langle 1+6t, 2+3t, 3+3t \rangle$, $0 \le t \le 1$ ▫

(b) the curve of intersection of the cylinder $x^2+y^2=4$ and the plane $2x+3y+z=1$.

- First note that projection of this curve to xy-plane is a circle given by eq-n $x^2+y^2=4 \Rightarrow$ parametrized via $\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$, $0 \le t \le 2\pi$

  Find z-component via $z = 1-2x-3y = 1-4\cos t - 6\sin t$.

  So: $\vec{r}(t) = \langle 2\cos t, 2\sin t, 1-4\cos t-6\sin t \rangle$ ▫

Ex6: Hand out worksheet with matching game

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* Derivatives (of vector f-s)

Formal definition of the derivative $\vec{r}\,'(t)$ of a vector f-n $\vec{r}(t)$ is given analogously to the $\mathbb{R}$-valued f-n:

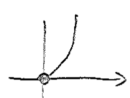
$$\boxed{\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}}$$

However, this is not the way we are gonna compute derivatives.
If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then clearly derivative is component-wise:

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}}$$

Geometrical meaning: 1) The vector $\vec{r}\,'(t)$ is tangent to the space curve determined by $\vec{r}$ at the point $\vec{r}(t)$.
2) If we think of $\vec{r}(t)$ as a coordinate of the particle moving along the trajectory C, then $|\vec{r}\,'(t)|$ = speed of particle.

Ex 7: (a) Sketch the plane curve with parametric eq-n $x = e^t, y = 2e^{3t}$.
▸ $y = 2x^3$, $x > 0$



∎

(b) Find $\vec{r}\,'(t)$
▸ $\vec{r}\,'(t) = \langle e^t, 6e^{3t}\rangle$ ∎

(c) Find the unit tangent vector at the point with $t=0$.
▸ $\vec{r}\,'(0) = \langle e^0, 6e^0\rangle = \langle 1, 6\rangle \Rightarrow T(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{1}{\sqrt{37}}\vec{i} + \frac{6}{\sqrt{37}}\vec{j}$ ∎

(d) Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}\,'(t)$ for $t=0$.

! There is a bunch of properties - see p. 858 of textbook.
The most interesting are:

$$\boxed{\begin{aligned}(\vec{u}(t)\cdot\vec{v}(t))' &= \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)\\ (\vec{u}(t)\times\vec{v}(t))' &= \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)\end{aligned}}$$

Ex 8: Find the parametric eq-n for the tangent line to the helix $x = 2\cos t$, $y = 2\sin t$, $z = t$ at the point $(0, 2, \frac{\pi}{2})$.

▶ This point corresponds to $t = \pi/2$.

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 1\rangle \Rightarrow \vec{r}\,'(\frac{\pi}{2}) = \langle -2, 0, 1\rangle$.

Hence: tangent line is given by $\underset{= -2t}{x = 0 - 2t}$, $y = 2 + 0\cdot t = 2$, $z = \frac{\pi}{2} + t$

Note: you better use another letter instead of t ■

* Integrals (of vector f-s)

Integrals are also defined component-wise. If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\boxed{\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle}$$

Ex 9: Evaluate $\int_0^1 (te^{t^2}\vec{i} + 2\sin t\cos t\,\vec{j} + e^{5t}\vec{k})\,dt$.

▶ $\int_0^1 te^{t^2}dt = \frac{e^{t^2}}{2}\Big|_0^1 = \frac{e-1}{2}$; $\int_0^1 2\sin t\cos t\,dt = \sin^2 t\Big|_0^1 = \sin^2(1)$; $\int_0^1 e^{5t}dt = \frac{e^{5t}}{5}\Big|_0^1 = \frac{e^5-1}{5}$

So: get $\frac{e-1}{2}\vec{i} + \sin^2(1)\cdot\vec{j} + \frac{e^5-1}{5}\cdot\vec{k}$ ■

* Length of a curve

For the case of plane curves given by $x = f(t)$, $y = g(t)$, many of you know that the length of the corresponding curve for $a \le t \le b$ equals

$$\boxed{L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt}$$

Likewise, given a space curve $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \le t \le b$, its length equals

$$\boxed{L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt = \int_a^b |\vec{r}\,'(t)|\,dt}$$

Rmks: 1) Explain roughly where this formula is coming from.

2) Note that we should request that the curve is traversed exactly once

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Ex 10: Find the length of the curve given by
$$\vec{r}(t) = 3\cos t \cdot \vec{i} + 3\sin t\, \vec{j} + 4t \cdot \vec{k} \quad \text{from } (3,0,0) \text{ to } (3,0,16\pi)$$

These points correspond to $t=0$ and $t=4\pi$.
$$L = \int_0^{4\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}\, dt = \int_0^{4\pi} 5\, dt = 20\pi.$$

* Velocity & Acceleration

Imagine a particle moving in $\mathbb{R}^3$ with position vector $\vec{r}(t)$. Then the velocity vector is the limit of the displacement vector and equals
$$\boxed{\vec{v}(t) = \vec{r}\,'(t)} \leftarrow \text{rate of change of distance w.r.t. time.}$$

The speed of a particle at time t equals $|\vec{v}(t)| = |\vec{r}\,'(t)|$

The acceleration of the particle is defined as the derivative of velocity
$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

Ex 11: A moving particle starts at the initial position $\vec{r}(0) = \langle 2,0,3\rangle$, with initial velocity $\vec{v}(0) = \langle 1,2,-3\rangle$ and acceleration $\vec{a}(t) = \langle e^t, -2t, 2\sin t\rangle$. Find its velocity & position at time t.

$$\vec{a}(t) = \vec{v}\,'(t) \Rightarrow \vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du = \langle 1,2,-3\rangle + \int_0^t \langle e^u, -2u, 2\sin u\rangle du$$
$$= \langle 1,2,-3\rangle + \langle e^t - 1, -t^2, -2\cos t + 2\rangle = \boxed{\langle e^t, 2-t^2, -1-2\cos t\rangle}$$
$$\vec{v}(t) = \vec{r}\,'(t) \Rightarrow \vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du = \langle 2,0,3\rangle + \int_0^t \langle e^u, 2-u^2, -1-2\cos u\rangle du$$
$$= \langle 2,0,3\rangle + \langle e^t - 1, 2t - \tfrac{t^3}{3}, -t - 2\sin t\rangle = \boxed{\langle e^t+1, 2t - \tfrac{t^3}{3}, 3-t-2\sin t\rangle}$$

So:
$$\boxed{\begin{aligned} \vec{v}(t) &= \langle e^t, 2-t^2, -1-2\cos t\rangle \\ \vec{r}(t) &= \langle e^t+1, 2t-\tfrac{t^3}{3}, 3-t-2\sin t\rangle \end{aligned}}$$