

* Organizational

- office hours
- Peer tutors
- hand out homework #1.

* Last time

Discussed:

- vector functions and space curves
- limits, derivatives, integrals of vector functions
- length of curves

* Velocity & Acceleration - see last page from Lecture #4 Notes

* Functions of several variables - main topic for today.

Def: A function of two variables is a function whose domain D is a subset of \mathbb{R}^2 and whose range is a subset of \mathbb{R} .

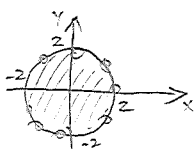
We will often write $z = f(x, y)$ (but you can use any other letters!)

Convention: If a function f is given by the formula and no domain is specified, then the domain of f is the set of all points $(x, y) \in \mathbb{R}^2$ such that $f(x, y)$ is well-defined, while the range of f is the set of all possible values $f(x, y)$.

Ex 1: Sketch the domain for each of the following functions:

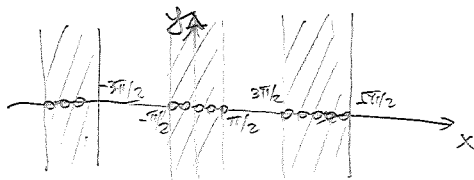
(a) $f(x, y) = \sqrt{y - x^2}$

(b) $f(x, y) = \ln(4 - x^2 - y^2)$



boundary excluded!

(c) $f(x, y) = \frac{\sqrt{\cos x}}{y}$



Ex 2: Find the range of all 3 functions in Ex 1.

- (a) $[0, +\infty)$
- (b) $(-\infty, \ln 4]$
- (c) $\mathbb{R} = (-\infty, \infty)$

Provide arguments!

* Graphs and level curves

The most common way to visualize a function of two variables is by considering the corresponding graph:

Def: If f is a function of two variables with domain D , then the graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ s.t. $(x, y) \in D$ & $z = f(x, y)$

Note: This is in a complete analogy by drawing a graph of $y = f(x)$.

An alternative way to visualize $f(x, y)$ is by drawing level curves:

Def: The level curves of a function f of two variables are the curves with equation $f(x, y) = k$, where k is a constant (in the range of f)

Ex 3: Sketch the graphs of the following functions:

- (a) $f(x, y) = 1 - x - y$
- (b) $f(x, y) = \sqrt{9 - x^2 - y^2}$
- (c) $f(x, y) = x^2 + 9y^2$
- (d) $f(x, y) = \sqrt{4 - y^2}$
- (e) $f(x, y) = 3\sqrt{x^2 + y^2}$

(a) $z = 1 - x - y \Leftrightarrow x + y + z = 1$ - plane.
Intercepts are $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

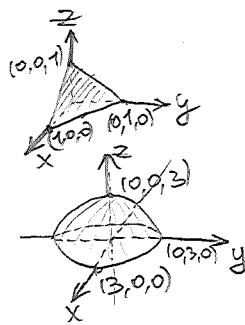
(b) $z = \sqrt{9 - x^2 - y^2} \Leftrightarrow \begin{cases} x^2 + y^2 + z^2 = 9 \\ z \geq 0 \end{cases}$ - half sphere



elliptic paraboloid

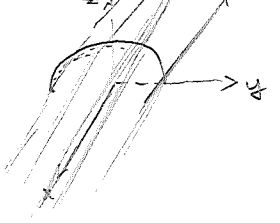
(c) $z = x^2 + 9y^2$

Note that intersecting with xz -plane get parabola $z = x^2$
 yz -plane get parabola $z = 9y^2$
 any plane $z = k > 0$ get ellipses



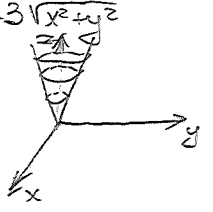
Lecture #5

(d) $z = \sqrt{4-y^2} \Leftrightarrow \begin{cases} y^2+z^2=4 \\ z \geq 0 \end{cases} \leftarrow \text{independent of } x.$



- half of cylinder

(e) $z = 3\sqrt{x^2+y^2}$

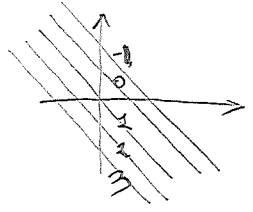


← cone

intersections with xz -plane - 2 halves of lines
 yz -plane - 2 halves of lines
 plane $z=k>0$ - circles

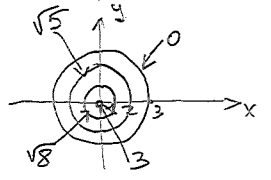
Ex 4: Draw the level curves in each of the above 5 cases.

(a) Level curves are given by solving $k = 1-x-y \Leftrightarrow x+y = 1-k$.



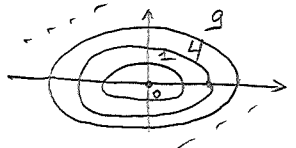
! The numbers over the level curves keep track of the level k .

(b) Level curves: $k = \sqrt{9-x^2-y^2} \Leftrightarrow \begin{cases} x^2+y^2 = 9-k^2 \\ 0 \leq k \end{cases}$



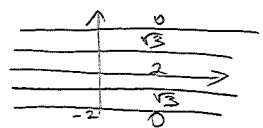
(circles)

(c) Level curves: $k = x^2+9y^2$. Clearly $k \in [0, \infty)$



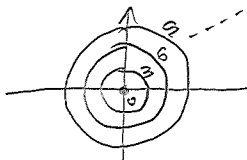
(ellipses)

(d) Level curves: $k = \sqrt{4-y^2} \Leftrightarrow \begin{cases} y = \pm \sqrt{4-k^2} \\ k \geq 0 \end{cases}$



(lines)

(e) Level curves: $k = 3\sqrt{x^2+y^2} \Leftrightarrow \begin{cases} x^2+y^2 = \frac{k^2}{9} \\ k \geq 0 \end{cases}$



(circles) but unbounded unlike n.(b)

- Remarks: (a) Do not forget to put numbers over level curves.
 (b) By analogy with topographic maps of mountains, we see that the corresponding graph is steep when the level curves are close enough, while it is flatter if they are further apart.

Ex 5: (a) How should we modify function $f(x,y)$ to flip upside down the corresponding graph?

(b) $-f$ —

to make it 3 times steeper?

(c) $-f$ —

to move it from the origin

to a point $(-1,3)$

(a) $f(x,y) \rightsquigarrow -f(x,y)$

(b) $f(x,y) \rightsquigarrow 3f(x,y)$

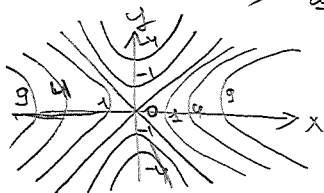
(c) $f(x,y) \rightsquigarrow f(x+1, y-3)$

Ex 6: Draw the level curves of the function $f(x,y) = x^2 - y^2$.

Level curves: $k = x^2 - y^2$

If $k=0$, this gives two lines $y = \pm x$.

If $k \neq 0$, rewrite this as $k = (x-y)(x+y) \Rightarrow$ get rotated hyperbola.



! The graph of this function is called hyperbolic paraboloid.

The reason for such a name is that intersecting with yz -plane, we obtain parabola $z = -y^2$ (looking downwards), with xz -plane get parabola $z = x^2$ (looking upwards), with planes $z = k \neq 0$ get hyperbolas.

Ex 7: Hand out matching game!