Lecture #6

09/18/2018

* Last time: functions of several variables and their visualizations

- Draw graphs and level curves to parts d,e of last exercise
- Last page of Notes from last time
- Emphasize that level curves are easier than graphs!

* Today: Sections 14.3 & 14.5 of your textbook (skipping 14.2!)

• Partial derivatives

Given a function of two variables, one can consider partial derivatives at a point (a,b), denoted by $f_x(a,b)$ and $f_y(a,b)$:

$$\boxed{f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}, \quad f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}}$$

Explicitly, to compute $f_x(a,b)$, we set $y=b$ to obtain a function of 1 variable $g(x) := f(x,b)$, and then take the usual derivative of $g(x)$ w.r.t. x at a.
Likewise, to compute $f_y(a,b)$, set $h(y) := f(a,y)$ and compute $h'(y)|_{y=b}$.

Ex1: If $f(x,y) = \cos(x^2y) + e^{x^2y^3} + y$, find $f_x(1,0)$ and $f_y(1,0)$

$f_x(x,y) = -\sin(x^2y)\cdot 2xy + e^{x^2y^3}\cdot 2xy^3 \Rightarrow f_x(1,0) = 0$

$f_y(x,y) = -\sin(x^2y)\cdot x^2 + e^{x^2y^3}\cdot 3x^2y^2 + 1 \Rightarrow f_y(1,0) = 1$

Ex2: If $f(x,y,z) = z^2\cos(x+e^y) + e^{2x}\sqrt{y^2+z^4}$, find $f_x(1,1,1)$, $f_y(1,1,1)$, $f_z(1,1,1)$.

! Here we note that partial derivatives are defined completely analogously by fixing all coordinates, except for the one we differentiate w.r.t.

• Geometric interpretation

If we intersect the graph of the function $f(x,y)$ with the plane $y=b$, then inside this plane we get a graph of the f-n $g(x)$, therefore, $f_x(a,b) = g'_x(a)$ is a slope of the tangent line to this graph at $x=a$.

Likewise, intersecting the graph with the plane $x=a$, we get a graph of the f-n $h(y)$, hence, $f_y(a,b) = h'_y(b)$ is a slope of the tangent line to this graph at $y=b$.

Lecture #6 09/18/2018

• Higher Derivatives

If f is a function of two variables, then f_x and f_y are also f-s of two variables, so we can consider their partial derivatives as well:

$(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$ ← second partial derivatives of f

$$\frac{\partial^2 f}{\partial x^2} \quad \frac{\partial^2 f}{\partial y \partial x} \quad \frac{\partial^2 f}{\partial x \partial y} \quad \frac{\partial^2 f}{\partial y^2}$$

Ex3: Find the second partial derivatives of $f(x,y) = \cos(xe^y)$

$f_x(x,y) = -\sin(xe^y)\cdot e^y$

$f_y(x,y) = -\sin(xe^y)\cdot xe^y$

$f_{xx}(x,y) = -\cos(xe^y)\cdot e^y \cdot e^y = -\cos(xe^y)\cdot e^{2y}$

$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(-\sin(xe^y)\cdot e^y\right) = -\cos(xe^y)\cdot x\cdot e^{2y} - \sin(xe^y)\cdot e^y$

$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(-\sin(xe^y)\cdot xe^y\right) = -\cos(xe^y)\cdot x\cdot e^{2y} - \sin(xe^y)\cdot e^y$

$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(-\sin(xe^y)\cdot xe^y\right) = -\cos(xe^y)\cdot x^2 e^{2y} - \sin(xe^y)\cdot xe^y$ ■

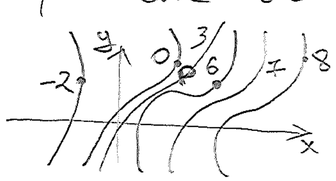
! Note that $f_{xy} = f_{yx}$ above.

Theorem: If f_{xy}, f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Ex4: Verify that the function $u(x,t) = \sin(x-at)$ satisfies the wave equation $u_{tt} = a^2\cdot u_{xx}$

$$\left.\begin{array}{l} u_{tt} = -a^2 \sin(x-at) \\ u_{xx} = -\sin(x-at \end{array}\right\} \Rightarrow u_{tt} = a^2\cdot u_{xx}$$ ■

Ex5: Given level curves of f below determine if f_x, f_y are positive or negative at pt P.



Lecture #6 09/18/2018

· Chain Rule

Let us start by recalling the chain rule from high school

$y = f(x),\ x = g(t) \Rightarrow y = f(g(t))$ – representing y as a function of t.

Then: $\boxed{\frac{d}{dt}y = \frac{dy}{dx}\cdot\frac{dx}{dt} = f'(g(t))\cdot g'(t)}$ $\leftarrow$ Ex: $y = e^{t^2}$.

Now, we want an analogous rule in the case when each function involves several variables.

Case 1: $z = f(x,y)$, $x = g(t)$, $y = h(t) \Rightarrow z = f(g(t), h(t))$ – function of t.

$$\boxed{\frac{dz}{dt} = f_x(g(t),h(t))\cdot g'(t) + f_y(g(t),h(t))\cdot h'(t) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}}$$

Ex6: If $z = x^2 + e^{3y}$, $x = \sin(t^2)$, $y = \cos(t^3)$, find $\left.\frac{dz}{dt}\right|_{t=0}$.

▶ One way: express z in terms of t and then differentiate

$z = \sin(t^2)^2 + e^{3\cos(t^3)} \Rightarrow \frac{dz}{dt} = 2\cdot\sin(t^2)\cdot\cos(t^2)\cdot 2t + e^{3\cos(t^3)}\cdot(-3\sin(t^3))\cdot 3t^2$

$\Rightarrow \left.\frac{dz}{dt}\right|_{t=0} = 0.$

Second way: $\frac{dz}{dt} = \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt} = 2\cdot\sin(t^2)\cdot\cos(t^2)\cdot 2t$
$+ e^{3\cos(t^3)}\cdot(-3\sin(t^3))\cdot 3t^2$ – same expression ∎

Case 2: $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t) \Rightarrow z = f(g(s,t), h(s,t))$ – function of s,t.

$$\boxed{\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\end{aligned}}$$

Ex7: If $z = \cos x\cdot e^y$, $x = s^2 + t$, $y = st$, find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$.

▶ $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = -\sin(x)\cdot e^y\cdot 2s + \cos(x)\cdot e^y\cdot t$

$= -\sin(s^2+t)\cdot e^{st}\cdot 2(s^2+t) + \cos(s^2+t)\cdot e^{st}\cdot t$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = -\sin(x)\cdot e^y\cdot 1 + \cos(x)\cdot e^y\cdot s$

$= -\sin(s^2+t)\cdot e^{st} + \cos(s^2+t)\cdot e^{st}\cdot s$ ∎

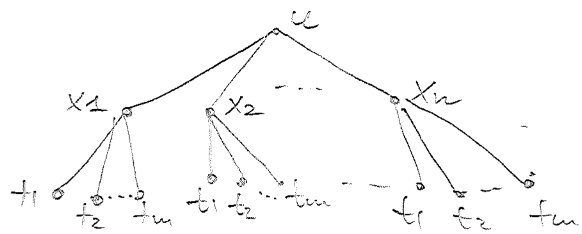
Lecture #6 09/18/2018

Case 3: Let $u = u(x_1, \to x_n)$, $x_1 = f_1(t_1, \to t_m), \dots, x_n = f_n(t_1, \to t_m)$.

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}}$$

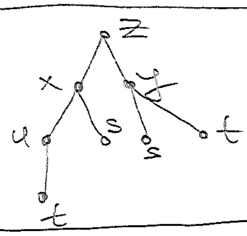
Ex 8: If $u = x^2 y + y^2 z^3$, while $x = r e^{s^2}$, $y = \sin(r+s) \cdot t^2$, $z = e^{r-s} \cdot \cos(t)$

Find the value $\frac{\partial u}{\partial s}$ when $r = 1, s = 0, t = 1$.



associated tree

Ex 9: If $z = e^x \sin(y)$, $x = u^2 s$, $y = s^2 t^3$, $u = e^t$, find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$.



associated tree

$$\frac{\partial z}{\partial s} = e^{e^{2t} \cdot s} \cdot \sin(s^2 t^3) \cdot e^{2t} + e^{e^{2t} \cdot s} \cdot \cos(s^2 t^3) \cdot 2 s t^3$$

$$\frac{\partial z}{\partial t} = e^{e^{2t} \cdot s} \cdot \sin(s^2 t^3) \cdot 2 e^t s \cdot e^t + e^{e^{2t} \cdot s} \cdot \cos(s^2 t^3) \cdot 3 s^2 t^2$$

! Explain how associated tree allows easily to write down the answer.

Lecture #6 09/18/2018

• Implicit differentiation

Suppose $\underline{y \text{ is a function in } x}$ ("$y = f(x)$"), which we do not know explicitly, but rather know implicitly via $F(x,y)=0$ for a given $F(\cdot,\cdot)$.

Then $0 = F(x,y) = F(x, f(x)) \Rightarrow 0 = \frac{d}{dx} F(x, f(x)) = \frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial y} \cdot \underbrace{f'(x)}_{\text{"}\partial y/\partial x\text{"}}$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \Rightarrow \boxed{y' = -\frac{F_x}{F_y}}$$

Ex 10 : Find y' if $x^3 + e^{2y} = 2x^2y^2$

This is equivalent to $F(x,y)=0$, where $F(x,y) = x^3 + e^{2y} - 2x^2y^2$.

So: $\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{3x^2 - 4xy^2}{2e^{2y} - 4x^2y}$

Likewise, if $z = f(x,y)$ and f is not given explicitly, but rather we have an implicit relation $F(x,y,z)=0$, then we can compute $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ in the same fashion.

$$\left.\begin{aligned} 0 &= \frac{\partial}{\partial x}\big(F(x,y,f(x,y))\big) = \frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot 0 + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} \\ 0 &= \frac{\partial}{\partial y}\big(F(x,y,f(x,y))\big) = \frac{\partial F}{\partial x}\cdot 0 + \frac{\partial F}{\partial y}\cdot 1 + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

Ex 11 : Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ if $x^3 + y + z^4 = 2 - 3\cos(2x)e^{3y} \cdot z$

Rewrite as $F(x,y,z) = 0$ with $F(x,y,z) = x^3 + y + z^4 - 2 + 3\cos(2x)e^{3y}z$.

$$\frac{\partial z}{\partial x} = -\frac{3x^2 - 6\sin(2x)e^{3y}z}{4z^3 + 3\cos(2x)e^{3y}}$$

$$\frac{\partial z}{\partial y} = -\frac{1 + 9\cos(2x)e^{3y}z}{4z^3 + 3\cos(2x)e^{3y}}$$