Lecture #7 09/20/2018

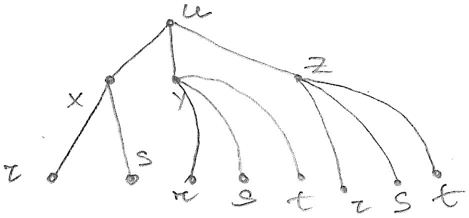
* **Organizational**

- Comments on homework 2 + Comments on homework 3: function $\Leftrightarrow$ level curves $\Leftrightarrow$ graph
- Change of office hours on Wed: $1^{30} - 2^{30}$.

* Last time: partial derivatives and chain rule.

Ex1: Verify that $u(x,t) = \sin(x - at)$ satisfies $u_{tt} = a^2 \cdot u_{xx}$

Ex2: If $u = x^2 y + y^2 z^3$, while $x = re^s$, $y = \sin(r+s)t^2$, $z = e^{r-s} \cdot \cos t$.
Find the value $\frac{\partial u}{\partial t}$ when $r = 1$, $s = 0$, $t = 0$.



associated graph

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t} =$$

$$= (x^2 + 2yz^3) \cdot (2t \cdot \sin(r+s)) + (3y^2 z^2) \cdot (-e^{r-s} \cdot \sin t)$$

$r = 1, s = 0, t = 0 \Rightarrow x = 1, y = \sin(1) \cdot 0 = 0, z = e^1 \cdot 1 = e \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ z = e \end{cases}$

So: $\left. \frac{\partial u}{\partial t} \right|_{r=1, s=0, t=0} = 0 + 0 = 0$

* Implicit differentiation

! Use material from p. 5 of Lecture #6 Notes – with two exercises.

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* Today: Directional derivatives (Section 14.6 of your textbook)

Last time we learnt $f_x(x_0, y_0)$ - rate of change of f as we move parallely to x-axis
$f_y(x_0, y_0)$ - rate of change of f as we move parallely to y-axis

Question: Given a direction $\vec{v}$ compute how fast f changes when we move in this direction?

For example, given a direction vector $\langle 3, 4\rangle$, take a unit vector $\vec{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$ in this direction; then moving by $\vec{u}$ is the same as moving by $\frac{3}{5}$ along x-axis and then by $\frac{4}{5}$ along y-axis. Therefore, it is kind of clear that the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\langle \frac{3}{5}, \frac{4}{5}\rangle$ equals $\frac{3}{5} f_x(x_0, y_0) + \frac{4}{5} f_y(x_0, y_0)$.

Let us give a formal definition:

Def: The directional derivative of $f(\cdot,\cdot)$ in the direction of a unit vector $\vec{u} = \langle a, b\rangle$ is

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}}$$

By what we discussed above, we arrive at the following formula:

$$\boxed{D_{\vec{u}} f(x, y) = f_x(x, y) \cdot a + f_y(x, y) \cdot b}$$ ← follows from the chain rule!

Note: If $\vec{u} = \vec{i}$, then $D_{\vec{i}} f = f_x$
$\vec{u} = \vec{j}$, then $D_{\vec{j}} f = f_y$

Geometrically: Intersect the graph of f with the plane parallel to z-axis and $\vec{u}$. Then, the slope of the tangent line to the resulting curve in that plane is exactly $D_{\vec{u}} f$

Ex3: Find the directional derivative of $f(x,y) = e^x \sin y$ at the point $(0, \frac{\pi}{2})$ in the direction indicated by the angle $\theta = -\frac{\pi}{4}$

▶ $\theta = -\pi/4 \Rightarrow \vec{u} = \langle \cos(-\frac{\pi}{4}), \sin(-\pi/4) \rangle = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$

$\partial_x f(x,y) = e^x \sin y \Rightarrow f_x(0, \frac{\pi}{2}) = 1$

$\partial_y f(x,y) = e^x \cos y \Rightarrow f_y(0, \frac{\pi}{2}) = 0$

So: $D_{\vec{u}} f(x,y) = 1/\sqrt{2} \cdot 1 + (-1/\sqrt{2}) \cdot 0 = 1/\sqrt{2}$ ∎

* Gradient Vector

Def: If f is a function of two variables x, y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

Note: $\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}}$

Ex4: Find the gradient of $f = \frac{x^2}{y^3}$. Find the rate of change of f at the point $P(3,1)$ in the direction of $\vec{u} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$

▶ $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle \frac{2x}{y^3}, -\frac{3x^2}{y^4} \rangle$

$D_{\vec{u}} f(3,1) = \nabla f(3,1) \cdot \vec{u} = \langle 6, -27 \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle = \frac{18+108}{5} = \frac{126}{5}$ ∎

* Generalization to functions of three variables

Def: For a f-n $f(\cdot,\cdot,\cdot)$ of 3 variables, the gradient of f denoted ∇f is

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \cdot \vec{i} + f_y \cdot \vec{j} + f_z \cdot \vec{k}$$

The directional derivative may be formally defined as before, and we get

$$\boxed{D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}}$$

Ex 5: Find the gradient of $f(x,y,z) = e^{2x}\sin(3yz)$
Find the directional derivative of f at $P(1,1,1)$ in the direction of the vector $\vec{v} = \langle 2,2,1\rangle$.

► $\nabla f(x,y,z) = \langle 2e^{2x}\sin(3yz),\ 3ze^{2x}\cos(3yz),\ 3ye^{2x}\cos(3yz)\rangle$

$\vec{u} = \vec{v}/|\vec{v}| = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$

$\Rightarrow D_{\vec{u}} f(1,1,1) = \langle 2e^2\sin(3),\ 3e^2\cos(3),\ 3e^2\cos(3)\rangle \cdot \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$

$= \left(\frac{4}{3}\sin(3) + 3\cos(3)\right)e^2$ ◻

* Max/Min of $D_{\vec{u}} f$

Recall $D_{\vec{u}} f = \nabla f \cdot \vec{u}$. Hence, given point (x_0, y_0, z_0) we have

$$\boxed{-|\nabla f(x_0,y_0,z_0)| \le D_{\vec{u}} f(x_0,y_0,z_0) \le |\nabla f(x_0,y_0,z_0)|}$$

due to the geometric meaning of dot-product

equality iff $\vec{u}$ is in the opposite direction of $\nabla f(x_0,y_0,z_0)$

equality iff $\vec{u}$ is in the direction of $\nabla f(x_0,y_0,z_0)$

Ex 6: Consider $f(x,y,z) = e^{2x}\sin(3yz)$ (as in Ex 5) and $P=(1,1,1)$.
In what direction does f have the max/min directional derivative? What are the corresponding values?

* Tangent planes to level surfaces

Let S be a level surface of $F(x,y,z)$, i.e. S consists of all (x,y,z) such that $F(x,y,z) = k$ ← fixed number. And let $P(x_0,y_0,z_0)$ be a point on S. ($\Rightarrow k = F(x_0,y_0,z_0)$). The tangent plane to S at point P is a plane passing through P and perpendicular to $\nabla F(x_0,y_0,z_0)$. Explicitly, it is given by the equation

$$\boxed{F_x(x_0,y_0,z_0)\cdot(x-x_0) + F_y(x_0,y_0,z_0)\cdot(y-y_0) + F_z(x_0,y_0,z_0)\cdot(z-z_0) = 0}$$

The key property of the tangent plane is that if we consider any curve C on S passing through P_0, then the tangent vector to this curve at P_0 lies in the tangent plane

Def: The normal line to the level surface S at P is the line through P and perpendicular to the tangent plane, i.e. in the direction of $\nabla F(x_0, y_0, z_0)$. Here if $F_x \neq 0, F_y \neq 0, F_z \neq 0$ at (x_0, y_0, z_0), then this line is given by

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Key Example: Surface S is of the form $z = f(x,y)$ (S - graph of f)

$\Rightarrow$ we can realize S as a level zero surface of the f-n

$$F(x,y,z) = f(x,y) - z.$$

Then: $\boxed{\nabla F(x_0, y_0, z_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle}$

$\Downarrow$

$$\boxed{\text{tangent plane}: \quad f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0) - (z - z_0) = 0}$$

Ex 7: Find equation of the tangent plane and the normal line to the given surface at the given point:

(a) $z^2 - 2xy + e^x = 5$, $P(0,1,2)$

(b) $z = e^{\sin(x + y^2)}$, $P(\pi, 0, 1)$

Ex 8: In the setup of Ex 7(a), at which point on this surface is the tangent plane parallel to the xy-plane?