Lecture #8

09/25/2018

* **Last time**
  - Directional derivative (remind meaning behind) $D_{\vec{u}} f$
  - Gradient vector ∇f (components are just partial derivatives)

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

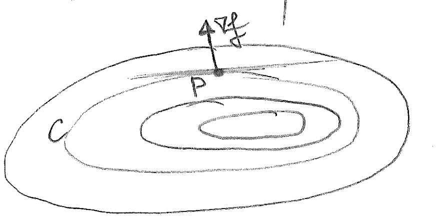
  - Maximizing / Minimazing $D_{\vec{u}} f$ (given f and a point P)

$\text{Max} = |\nabla f|$ for $\vec{u} = \frac{\nabla f}{|\nabla f|}$

$\text{Min} = -|\nabla f|$ for $\vec{u} = -\frac{\nabla f}{|\nabla f|}$

* **Tangent lines to level curves and tangent planes to level surfaces**

Before going to level surfaces we ended up with on Thursday, let us start from a 2-dim analogue. Consider level curves of function $f(x,y)$. Pick one such level curve $C: \{f(x,y) = k\}$ and choose a point P on C.



Clearly as we move along C, the value of f stays constant. In particular, this means that $D_{\vec{u}} f(P) = 0$ if $\vec{u}$ is a tangent (to C at point P) unit vector.

But $0 = D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} \Rightarrow \vec{u}$ is perpendicular to $\nabla f(P)$.

Moral: At any point P of a level curve C of $f(x,y)$, the gradient vector $\nabla f(P)$ is perpendicular to the tangent line to C at P.

Consequence: Equation of the tangent line to C at $P(x_0, y_0)$ is

$$\boxed{f_x(x_0,y_0)\cdot(x - x_0) + f_y(x_0,y_0)\cdot(y - y_0) = 0}$$

Now: Remind 3d analogue from last time and do Ex 7, 8 from previous notes.

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09/25/2018

* New topic for the entire week is "Max/Min Problems" (Section 14.7)

The goal of today's and Thursday's discussion is to learn how to find max/min values of f-n on the plane or a certain domain $D \subseteq \mathbb{R}^2$.

Let us start by recalling 1-d analogue from high school:

Ex1: Find the (absolute) maximal & minimal value of $f(x)=x^3-12x$ on $[-3,5]$.

- Critical points: $0=f'(x)=3x^2-12 \Rightarrow x=\pm 2$ and $f(2)=-16$, $f(-2)=16$.
- Also check end-points: $f(-3)=9$, $f(5)=65$.

So: Max is 65 (achieved at $x=5$), Min is -16 (achieved at $x=2$) ∎

Def: (a) A function $f(x,y)$ has a local maximum at (a,b) if $f(x,y) \le f(a,b)$ for any (x,y) "near" (a,b). The number $f(a,b)$ is called a local max value.

(b) A function $f(x,y)$ has a local minimum at (a,b) if $f(x,y) \ge f(a,b)$ for any (x,y) "near" (a,b). The number $f(a,b)$ is called a local min value

(c) If the inequalities in (a) & (b) hold for any (x,y) in the domain, then (a,b) is called an absolute max value or absolute min value.

Note that considering $g(x):=f(x,b)$, $h(y):=f(a,y)$ – functions of 1 variable, we see that

$$\left.\begin{array}{l} x=a \text{ is a local max/min for } g(x) \\ y=b \text{ is a local max/min for } h(y) \end{array}\right\} \Rightarrow f_x(a,b)=f_y(a,b)=0$$

Theorem: If f has a local max or min at the point (a,b) and f_x, f_y – exist then $\boxed{f_x(a,b)=f_y(a,b)=0}$

Def: A point (a,b) in the domain is called critical of $f(x,y)$ if $f_x(a,b)=f_y(a,b)=0$ or f_x or f_y do not exist.

Warning: Local Max/Min $\Rightarrow$ critical, but Critical $\not\Rightarrow$ Local Max/Min.

Important Remark: Note that if $f_x(a,b)=0=f_y(a,b)$, then the tangent plane to the graph of $f(\cdot,\cdot)$ at the point $(a,b,f(a,b))$ is parallel to xy-plane.

Ex 2: Find $\underbrace{\text{local max/min}}_{\text{"extreme"}}$ values of $f(x,y)=x^2-y^2$.

$f_x = 2x$, $f_y = -2y \Rightarrow$ only one critical point: $(0,0)$

However, $f(x,0)=x^2>0$ as x approaches to 0

$f(0,y)=-y^2<0$ as y approaches to 0

Thus: f has no extreme values at all! ■

Theorem (Second Derivative Test): Suppose second partial derivatives of f are continuous near (a,b), and assume $f_x(a,b)=f_y(a,b)=0$.

Define

$$\boxed{D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - (f_{xy}(a,b))^2}$$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

(c) If $D<0$, then $f(a,b)$ is NOT a local max/min.

(d) If $D=0$, then nothing can be said

Def: If $D<0$ above, then (a,b) is called a saddle point of f

Ex 3: Find the local max/min values and saddle points of

(a) $f(x,y)=y\sin x$

(b) $f(x,y)=5-x^4+2x^2-y^2$.

(a) $f_x = y\cos x$, $f_y=\sin x \Rightarrow$ Critical points are $\{(\pi k, 0) \mid k\text{-integer}\}$.

$D = f_{xx}(\pi k,0)\, f_{yy}(\pi k,0) - (f_{xy}(\pi k,0))^2 = -1 < 0$

So: There are no local max/min values, while saddle points are $\{(\pi k,0) \mid k\text{- integer}\}$

(b) → next page

(b) $f_x(x,y) = -4x^3+4x = 4x(1-x)(1+x)$ – zero iff $x \in \{0,-1,1\}$
$f_y(x,y) = -2y$ – zero iff $y=0$.
$\Rightarrow$ Critical points: $(0,0)$, $(-1,0)$, $(1,0)$

$f_{xx}(x,y) = 4-12x^2$, $f_{yy}(x,y) = -2$, $f_{xy}(x,y)=0$. $\Rightarrow D = 2(12x^2-4)$

* At point $(0,0)$: $D=-8<0 \Rightarrow$ $(0,0)$ – saddle point
* At point $(-1,0)$: $D=16>0$, $f_{xx}(-1,0)=-8<0 \Rightarrow$ $(-1,0)$ – local max
* At point $(1,0)$: $D=16>0$, $f_{xx}(1,0)=-8<0 \Rightarrow$ $(1,0)$ – local max

$f(-1,0) = 6 = f(1,0)$

So: There are no local min, while local max: $(\pm 1,0)$ (values of f at these pts are 6)
saddle points: $(0,0)$ ∎

Ex 4: Find the point on the plane $x+y+5z-1=0$ that is closest to $P(1,2,5)$.

Distance $d = \sqrt{(x-1)^2+(y-2)^2+(z-5)^2}$

Want: Minimize d or equivalently d^2 (as $d \geq 0$ always)

Plane equation $\Rightarrow x = 1-y-5z \Rightarrow d^2 = \underbrace{(y-2)^2+(z-5)^2+(y+5z)^2}_{f(y,z)}$

$f_y(y,z) = 2(y-2)+2(y+5z) = 2(2y+5z-2)$

$f_z(y,z) = 2(z-5)+10(y+5z) = 10y+52z-10$

So: need to solve $\begin{cases} 2y+5z-2=0 \\ 10y+52z-10=0 \end{cases} \Rightarrow \begin{cases} z=0 \\ y=1 \end{cases} \Rightarrow x = 1-1-5\cdot 0 = 0$

Hence, there is only one critical point: $(0,1,0)$. From geometric interpretation it is clear that it must be exactly the closest point. But, let us check using 2nd derivative test

$f_{yy}(1,0)=4$, $f_{zz}(1,0)=52$, $f_{yz}=10 \Rightarrow D = 4\cdot 52-10^2>0 \xrightarrow{f_{yy}(1,0)>0} (y,z)=(1,0)$ – loc. min.
(geom. clear it is absolute min) ∎