Lecture #9 09/27/2018

* Last time
  - Critical points for $f(x,y)$
  - Second derivative test
    ↳ evaluate $D = f_{xx} f_{yy} - (f_{xy})^2$ for all critical points.

  ◦ Do 2 exercises from the end of last class!

* Today: Absolute Max/Min.

Let us start by recalling the situation for functions of 1 variable.

Ex 0: Find the absolute max and min values of the function x^3 on
(a) $\mathbb{R}$ ← no max/min
(b) $[-2,2]$ ← -8 & 8

This illustrates the general principle that $f(x)$ may have no absolute max/min on $\mathbb{R}$, but any continuous $f(x)$ achieves absolute max/min on any closed interval $[a,b]$

Ex 1: Find the absolute max and min of x^2-2x+y^2+2y+5 on $\mathbb{R}^2$

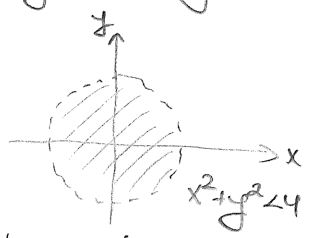
▸ $f(x,y) = x^2-2x+y^2+2y+5 = (x-1)^2+(y+1)^2+3$

No absolute max, while absolute minimum $= 3$ ∎

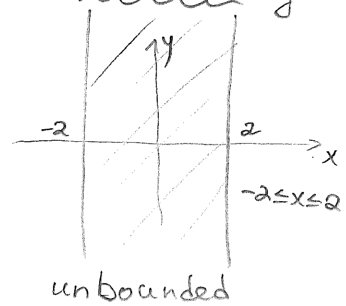
As for f-s of 1 variable, there may be no absolute max/min of $f(x,y)$ on $\mathbb{R}^2$. However, if f is continuous on a closed and bounded set $D \subseteq \mathbb{R}^2$, then a general theorem guarantees that absolute max/min are achieved on D.

Rmk: In the last exercice from last time, when computing a distance from a given pt to a given plane, geometry immediately "says" that absolute max is not achieved, abs. min is achieved

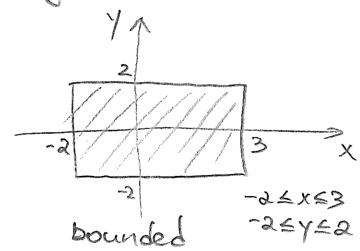
Def: A region $D \subseteq \mathbb{R}^2$ is <u>bounded</u> if it is not infinite.



bounded
not closed



unbounded
closed



bounded
closed

Def: A region $D \subseteq \mathbb{R}^2$ is <u>closed</u> if it includes its boundary

<u>Ex 2</u>: Are the following regions closed or/and bounded:

(a) $0 \le x \le 1$, $0 \le y \le 1-x$

(b) $0 \le y < \sqrt{9-x^2}$

(c) $x \le y \le x+10$.

> Algorithm to find absolute max/min on the closed & bounded $D \subseteq \mathbb{R}^2$
> - First, find the values of f at the critical points of f <u>inside</u> D
> - Second, find the extreme values of f on the boundary of D
> - The largest/smallest of the values in the above 2 steps is the absolute max/min of f on D

<u>Warning</u>: When find extreme values of f on the boundary of D, you <u>CAN NOT</u> apply the 2^{nd} Derivative test. Instead, you should split boundary into several pieces, each of which is easy to parametrize by 1 variable, and thus reduce the question to the case of functions of 1 variable

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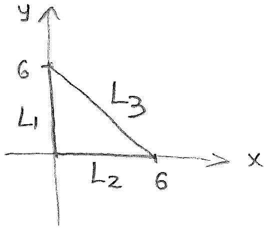
Ex 3: Find the absolute max / min of $f(x,y) = x^2 - 4x + y^2 - 6y + 1$ in the closed triangular region with vertices $(0,0)$, $(6,0)$, $(0,6)$.



Step 1: Critical pts

$$\left.\begin{array}{l} f_x(x,y) = 2x - 4 = 0 \text{ iff } x = 2 \\ f_y(x,y) = 2y - 6 = 0 \text{ iff } y = 3 \end{array}\right\} \Rightarrow \text{get only 1 critical pt } (2,3)$$

and $f(2,3) = \boxed{-12}$

Step 2: $\boxed{L_1 = \{(0,y) : 0 \le y \le 6\}}$

$g(y) = f(0,y) = y^2 - 6y + 1$

$g'(y) = 2y - 6 = 0$ iff $y = 3$ and $g(y) = \boxed{-8}$

Also compute at end-points: $g(0) = \boxed{1}$, $g(6) = \boxed{1}$

$\boxed{L_2 = \{(x,0) : 0 \le x \le 6\}}$

$g(x) = f(x,0) = x^2 - 4x + 1$

$g'(x) = 2x - 4 = 0$ iff $x = 2$: $g(2) = \boxed{-3}$

Also end-points: $g(0) = 1$, $g(6) = \boxed{13}$

↑ already listed above

$\boxed{L_3 = \{(x, 6-x) : 0 \le x \le 6\}}$

$g(x) = x^2 - 4x + (36 - 12x + x^2) - (36 - 6x) + 1 = 2x^2 - 10x + 1$

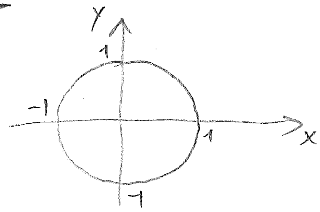
$g'(x) = 4x - 10 = 0$ iff $x = \frac{5}{2}$ and $g\left(\frac{5}{2}\right) = \frac{25}{2} - 25 + 1 = \boxed{-\frac{23}{2}}$

Step 3: Choose max & min value of the above listed:

Absolute max $= \underline{13}$ and it is achieved at $\underline{(6,0)}$

Absolute min $= \underline{-12}$ and it is achieved at $\underline{(2,3)}$

Ex 4: Find the absolute max/min of $f(x,y) = 2x^4 + y^4$ on $D: x^2+y^2 \le 1$.



Critical pts: $f_x = 8x^3 = 0$ iff $x=0$, $f_y = 4y^3 = 0$ iff $y=0$ $\Rightarrow$ critical points: $(0,0)$ and $f(0,0) = \boxed{0}$

Boundary: unit circle, which can be parametrized by $\{(\cos\theta, \sin\theta) \mid 0 \le \theta \le 2\pi\}$.

$g(\theta) = f(\cos\theta, \sin\theta) = 2\cos^4\theta + \sin^4\theta$

$g'(\theta) = -8\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta = 4\cos\theta\sin\theta(\sin^2\theta - 2\cos^2\theta) = 4\cos\theta\sin\theta(3\sin^2\theta - 2)$

So: $g'(\theta) = 0$ iff $\underbrace{\cos\theta = 0}_{\text{points } (0,1),(0,-1)}$, or $\underbrace{\sin\theta = 0}_{\text{points } (1,0),(-1,0)}$, or $\sin^2\theta = \frac{2}{3} \Rightarrow \cos^2\theta = \frac{1}{3}$

$f(0,1) = f(0,-1) = \boxed{1}$, $f(1,0) = f(-1,0) = \boxed{2}$

Finally, at points $(\cos\theta, \sin\theta)$ such that $\cos^2\theta = \frac{1}{3}$, $\sin^2\theta = \frac{2}{3}$, we have

$$f(\cos\theta, \sin\theta) = 2\cdot\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \boxed{\frac{6}{9}}$$

So: Absolute max $= 2$ and it is achieved at $(\pm 1, 0)$

Absolute min $= 0$ and it is achieved at $(0,0)$ $\blacksquare$

Ex 5: Find the global (= absolute) max and min of $f(x,y) = x^2y - 2xy + y^2$ on the region $0 \le y \le x$, $0 \le x \le 2$.

Ex 6: Find the global max and min of $f(x,y) = x^2 - y$ on the square $-1 \le x \le 1$, $-1 \le y \le 1$

Ex 7: Find the absolute max and min of $f(x,y) = xy$ on $D: x^2+y^2 \le 1$.