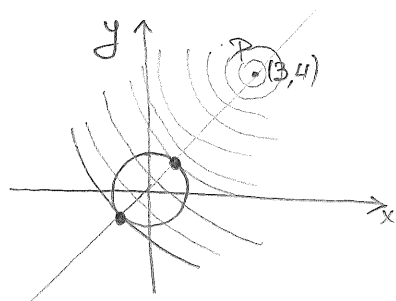


*Today: Lagrange Multipliers

Ex 1: Find a point on the unit circle $x^2 + y^2 = 1$ which is most closely located to the point $P(3, 4)$.



Geometric observation: Intuitively it is clear how to find such a point as it suffices to look at the intersections of the circle and the line connecting the center of the circle and P .

However, we want a rigorous approach.

Goal: Minimize $f(x,y) = (x-3)^2 + (y-4)^2$ with a constraint $g(x,y) = 1$
 $x^2 + y^2$

Key Observation: At the points where $f(x,y)$ is min/max, the tangent line to the level curve of f coincides with the tangent line of the curve given by $g(x,y) = 1$.

The latter implies that ∇f and ∇g are proportional at the given point. $\Rightarrow \nabla f(x,y) = \lambda \cdot \nabla g(x,y)$

$$\begin{aligned} \nabla f(x,y) &= \langle 2(x-3), 2(y-4) \rangle \\ \nabla g(x,y) &= \langle 2x, 2y \rangle \end{aligned} \Rightarrow \begin{cases} 2(x-3) = \lambda \cdot 2x \\ 2(y-4) = \lambda \cdot 2y \end{cases} \Rightarrow \begin{cases} (1-\lambda)x = 3 \\ (1-\lambda)y = 4 \end{cases} \Rightarrow \begin{cases} x = \frac{3}{1-\lambda} \\ y = \frac{4}{1-\lambda} \end{cases} \Rightarrow$$

But: we also have constraint $x^2 + y^2 = 1$

$$\Rightarrow \left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{4}{1-\lambda}\right)^2 = 1 \Rightarrow (1-\lambda)^2 = 25 \Rightarrow \begin{cases} \lambda = -4 \\ \lambda = 6 \end{cases}$$

$$\begin{aligned} \text{If } \lambda = -4 &\Rightarrow x = \frac{3}{5}, y = \frac{4}{5} \Rightarrow \text{get point } \left(\frac{3}{5}, \frac{4}{5}\right) \\ \text{If } \lambda = 6 &\Rightarrow x = -\frac{3}{5}, y = -\frac{4}{5} \Rightarrow \text{get point } \left(-\frac{3}{5}, -\frac{4}{5}\right) \end{aligned} \Rightarrow \begin{cases} \left(\frac{3}{5}, \frac{4}{5}\right) - \text{closest} \\ \text{while} \\ \left(-\frac{3}{5}, -\frac{4}{5}\right) - \text{the most distant} \end{cases}$$

Clearly $f\left(\frac{3}{5}, \frac{4}{5}\right) < f\left(-\frac{3}{5}, -\frac{4}{5}\right)$

Answer: $\left(\frac{3}{5}, \frac{4}{5}\right)$

Lecture #10

10/02/2018

The situation is the same for the functions of 3 variables. In other words, if we want to minimize/maximize $f(x,y,z)$ given a constraint $g(x,y,z) = k$, we again solve the following system

$$\begin{cases} \nabla f(x,y,z) = \lambda \cdot \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases} \Leftrightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = k \end{cases} \quad \text{Def: } \lambda \text{ is called a Lagrange multiplier}$$

Method: ① Find all values (x,y,z) such that there exists λ : $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and also $g(x,y,z) = k$.

② Among the values of f at the points obtained in ① choose the min/max - these are the min/max values of f .

- Rmks:
- 1) We assume $\nabla g \neq 0$ on the surface $g(x,y,z) = k$.
 - 2) We assume absolute min/max exist
 - 3) We don't actually need explicit values for λ .
 - 4) If f, g - functions of two variables - apply the same procedure.

Ex 2: Find the min/max values of $f(x,y,z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 12$

$$\begin{aligned} \nabla f(x,y,z) &= \langle yz, xz, xy \rangle, \quad \nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle \\ \begin{cases} yz = \lambda \cdot 2x & \cdot x \\ xz = \lambda \cdot 2y & \cdot y \\ xy = \lambda \cdot 2z & \cdot z \\ x^2 + y^2 + z^2 = 12 \end{cases} &\Rightarrow xyz = 2\lambda \cdot x^2 = 2\lambda y^2 = 2\lambda z^2 \end{aligned}$$

Case 1: $\lambda = 0 \Rightarrow yz = xz = xy = 0 \Rightarrow$ two of the three variables $\{x,y,z\}$ must be 0, while the third is forced to be $\pm\sqrt{12}$ by the condition $x^2 + y^2 + z^2 = 12$. At all these points: $\boxed{f=0}$

Case 2: $\lambda \neq 0 \Rightarrow x^2 = y^2 = z^2$

Combining with constraint $x^2 + y^2 + z^2 = 12 \Rightarrow x^2 = y^2 = z^2 = 4 \Rightarrow 8 \text{ points: } (\pm 2, \pm 2, \pm 2)$

At $(2, 2, 2), (2, -2, -2), (-2, 2, -2), (-2, -2, 2) : f = \boxed{8}$

$(2, 2, -2), (2, -2, 2), (-2, 2, 2), (-2, -2, -2) : f = \boxed{-8}$

Answer: The max of f is equal to 8 achieved at $(2, 2, 2), (2, -2, -2), (-2, 2, -2), (-2, -2, 2)$

The min of f is equal to -8 achieved at $(-2, -2, -2), (2, 2, -2), (2, -2, 2), (-2, 2, 2)$.

Ex 3: Find the extreme values of $f(x, y) = e^{xy}$ on the region $D \subseteq \mathbb{R}^2$ given by $D = \{(x, y) \mid x^2 + 4y^2 \leq 4\}$.

! This problem requires a combination of last lecture & today's lecture

Step 1 (Critical points): $f_x = ye^{xy} = 0 \text{ iff } y=0$
 $f_y = xe^{xy} = 0 \text{ iff } x=0$ } Crit. points: $(0, 0) \Rightarrow f(0, 0) = \boxed{1}$

Step 2 (Boundary: $\underbrace{x^2 + 4y^2}_{g(x, y)} = 4$)

$\nabla f(x, y) = \langle ye^{xy}, xe^{xy} \rangle, \nabla g(x, y) = \langle 2x, 8y \rangle$

$\begin{cases} ye^{xy} = \lambda \cdot 2x \\ xe^{xy} = \lambda \cdot 8y \\ x^2 + 4y^2 = 4 \end{cases} \Rightarrow \begin{cases} \cdot x \\ \cdot y \end{cases} \Rightarrow \lambda \cdot 2x^2 = \lambda \cdot 8y^2 = xy e^{xy}$
either $\lambda = 0 \Rightarrow x=0, y=0$ which is impossible as $x^2 + 4y^2 = 4$
or

$\lambda \neq 0 \Rightarrow 2x^2 = 8y^2 \Rightarrow x^2 = 4y^2 \Rightarrow x^2 = 4y^2 = 2$

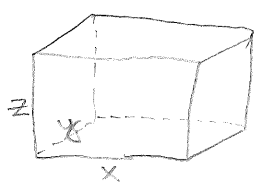
But combining this with $x^2 + 4y^2 = 4$

\Rightarrow get 4 points $(\sqrt{2}, \frac{1}{\sqrt{2}}), (-\sqrt{2}, -\frac{1}{\sqrt{2}}), (\sqrt{2}, -\frac{1}{\sqrt{2}}), (-\sqrt{2}, \frac{1}{\sqrt{2}})$ with values of $f: e, e^{-1}, e, e^{-1}$

As $\frac{1}{e} < 1 < e \Rightarrow$ Max. of f is e achieved at $(\sqrt{2}, \frac{1}{\sqrt{2}}), (-\sqrt{2}, -\frac{1}{\sqrt{2}})$

Min of f is e^{-1} achieved at $(\sqrt{2}, -\frac{1}{\sqrt{2}}), (-\sqrt{2}, \frac{1}{\sqrt{2}})$ (3)

Ex 4: A box is to be constructed with a volume of 500 cubic inches. The box has 4 sides and a bottom, but no top. What are the dimensions of the cheapest box?



Volume = $\underbrace{xyz}_{g(x,y,z)} = 500 \text{ inches}^3$

Surface Area = $\underbrace{xy + 2xz + 2yz}_{f(x,y,z)}$

$\nabla f(x,y,z) = \langle y+2z, x+2z, 2x+2y \rangle$

$\nabla g(x,y,z) = \langle yz, xz, xy \rangle$

$$\left. \begin{array}{l} y+2z = \lambda \cdot yz \\ x+2z = \lambda \cdot xz \\ 2x+2y = \lambda \cdot xy \\ xyz = 500 \end{array} \right\} \begin{array}{l} \cdot x \\ \cdot y \\ \cdot z \end{array} \Rightarrow \lambda xyz = xyz + 2xz = xy + 2yz = 2xz + 2yz$$

$\bullet \quad xy + 2xz = xyz + 2xz \Rightarrow z(x-y) = 0 \Rightarrow x=y$ as $z \neq 0$ due to $xyz=500$

$\bullet \quad xy + 2yz = 2xz + 2yz \Rightarrow x(y-2z) = 0 \Rightarrow y=2z$ as $x \neq 0$

So: $x=y=2z$.

Plugging this into $xyz=500 \Rightarrow 2z \cdot 2z \cdot z = 500 \Rightarrow z=5 \Rightarrow \left. \begin{array}{l} x=10 \\ y=10 \\ z=5 \end{array} \right\}$

As there is only 1 candidate $(x=10, y=10, z=5)$, it must be the solution!

Ex 5: Find min/max values of $\ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1)$ given $x^2+y^2+z^2=12$.

Ex 6*: (a) Maximize $x_1 + \dots + x_n$ subject to $x_1^2 + \dots + x_n^2 = 1$

(b) Maximize $\sqrt[n]{x_1 \cdot \dots \cdot x_n}$ subject to $\begin{cases} x_1 + \dots + x_n = 1 \\ x_1 \geq 0, \dots, x_n \geq 0 \end{cases}$