Lecture #11 10/04/2018

* Last time : Method of Lagrange Multipliers
  ↳ explain rigorously why we are solving $\nabla f = \lambda \cdot \nabla g$ (see page 972 of textbook)

* Today : Double integrals (Sections 15.1, 15.2)

Goal : Extend the familiar notion of $\int_a^b f(x)dx$ to the case of functions of two variables.

Warning : We are skipping the rigorous mathematical definition of double integrals - see pages 988-992 of your textbook.

Geometric Meaning : Double integrals can be utilized to compute "oriented" volumes under $z = f(x,y)$ in the same way usual integrals are used to compute "oriented" areas.

Iterated Integrals

Given a function $f(x,y)$ on the rectangle $[a,b]\times[c,d]$, we can evaluate two iterated integrals

$$\int_a^b \Big(\underbrace{\int_c^d f(x,y)\,dy}_{\substack{\text{here } x\text{-fixed} \\ \text{and we integrate} \\ \text{with respect to } y}}\Big)dx \quad \text{and} \quad \int_c^d \Big(\underbrace{\int_a^b f(x,y)\,dx}_{\substack{\text{here } y\text{-fixed} \\ \text{and we integrate} \\ \text{with respect to } x.}}\Big)dy$$

Ex1 : Evaluate the following iterated integrals:

(a) $\int_0^1 \int_0^2 xe^y dy\,dx \quad \left[= \int_0^1 \left(xe^y\Big|_{y=0}^{y=2}\right)dx = (e^2-1)\int_0^1 x\,dx = \frac{e^2-1}{2}\right]$

(b) $\int_0^2 \int_0^1 xe^y dx\,dy \quad \left[= \int_0^2 e^y \cdot \frac{1}{2}dy = \frac{e^2-1}{2}\right]$

Note : The answers to (a) and (b) coincide!

This is not a coincidence as the following Theorem says

Lecture #11 10/04/2018

**Theorem (Fubini's Theorem):** If $f(x,y)$ is continuous on the rectangle $R=[a,b]\times[c,d]=\{(x,y) \mid a\le x\le b \text{ and } c\le y\le d\}$, then

$$\boxed{\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d f(x,y)\,dx\,dy}$$

In particular, if $f(x,y)=g(x)\cdot h(y)$ on $R=[a,b]\times[c,d]$, then:

$$\boxed{\iint_R g(x)h(y)\,dx\,dy=\int_a^b g(x)\,dx\cdot\int_c^d h(y)\,dy}$$

**Ex2:** Evaluate $\iint_R 2ye^{xy}\,dA$, where $R=[0,1]\times[-1,1]$

$$\iint_R 2ye^{xy}\,dA=\int_{-1}^{1}\int_0^1 2ye^{xy}\,dx\,dy=\int_{-1}^{1}\left(2e^{xy}\Big|_{x=0}^{x=1}\right)dy=\int_{-1}^{1}2(e^y-1)\,dy=\boxed{2e-\frac{2}{e}-4}$$

**Ex3:** Find the volume of the solid $S$ that is bounded by the elliptic paraboloid $4x^2+y^2+z=10$, the planes $x=1$, $y=2$, and the three coordinate planes.

First note that $z=10-4x^2-y^2>0$ over the given region $[0,1]\times[0,2]$ in the $xy$-plane $\Rightarrow$ "oriented" volume $=$ usual volume.

$$\text{Vol}=\int_0^1\int_0^2(10-4x^2-y^2)\,dy\,dx=\int_0^1\left((10-4x^2)\cdot 2-\frac{8}{3}\right)dx=\int_0^1\left(\frac{52}{3}-8x^2\right)dx=\frac{52}{3}-\frac{8}{3}=\boxed{\frac{44}{3}}$$

**Practice Problems:**

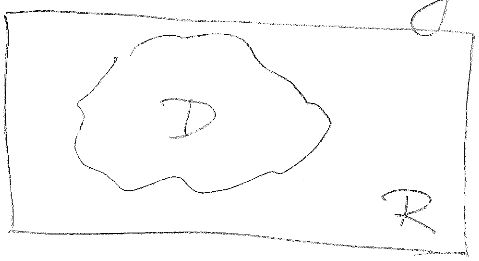
- Compute $\int_0^1\int_{-2}^{1}(y^2+y^3\sin x)\,dx\,dy$
- Compute $\iint_{[0,1]\times[0,2]}(ye^{xy}+x\sin(xy))\,dA$

Lecture #11 10/04/2018

## * Double Integrals over general Regions

In all previous problems the region over which we were integrating was a rectangle $R=[a,b]\times[c,d]$. However, there are way more regions in $\mathbb{R}^2$, so we need to integrate over all those.

Key Idea : Find a big enough rectangle $R=[a,b]\times[c,d]$ containing our region $D$ in $\mathbb{R}^2$



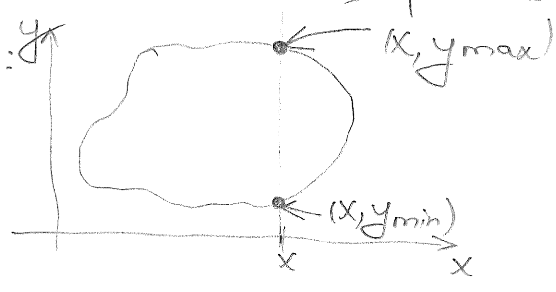

Consider a function $F(x,y)$ on $R$ given by:

$$\boxed{F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y)\in D \\ 0 & \text{if } (x,y)\in R\setminus D \end{cases}}$$

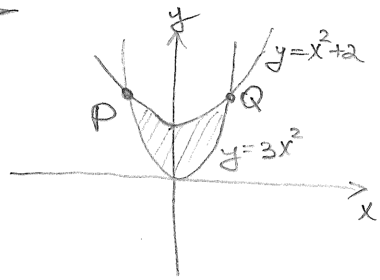
It is clear that the oriented volume under $F(x,y)$ over $R$ is the same as the volume under $f(x,y)$ over $D$. Hence, we define the double integral of $f(\cdot,\cdot)$ over $D$ via

$$\boxed{\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA}$$

In practice, this amounts to computing iterated integral, such that the limits of the (inner) integration are no longer fixed constants. E.g. if we use $\int\int\dots dy\,dx$, then the bounds for the inner integral are $\int_{y_{min}}^{y_{max}}\dots dy$, where:



**Ex 4**: Evaluate $\iint_D (x-y)\,dA$, where $D$ is the region bounded by $y=3x^2$, $y=x^2+2$



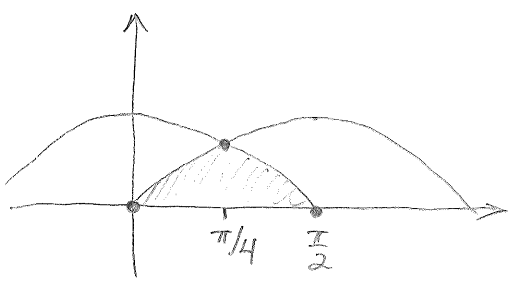

Solve $\underbrace{x^2+2=3x^2}_{x=-1 \text{ or } 1}$ to find x-coordinates of the intersection points P, Q.

$$\iint_D (x-y)\,dA = \int_{-1}^{1}\int_{3x^2}^{x^2+2} (x-y)\,dy\,dx = \int_{-1}^{1}\left[x\cdot(2-2x^2) - \frac{y^2}{2}\Big|_{y=3x^2}^{y=x^2+2}\right]dx$$

$$= \int_{-1}^{1}\left(2x-2x^3+\frac{9x^4-x^4-4x^2-4}{2}\right)dx = \int_{-1}^{1}(4x^4-2x^3-2x^2+2x-2)\,dx = \boxed{-\frac{56}{15}}$$

**Ex 5**: (a) Set up the integral $\iint_D (x-y)\,dA$, where $D$ - bounded by graphs of $y=\sin x$, $y=\cos x$, $0 \le x \le \frac{\pi}{2}$, $y \ge 0$

(b) Compute it.




$\sin x = \cos x \Leftrightarrow \tan x = 1 \xLongleftrightarrow{x\in[0,\frac{\pi}{2})} x = \frac{\pi}{4}$

But from the picture we see that we actually have to split this double integral into the sum of two:

$$\boxed{\iint_D (x-y)\,dA = \underbrace{\int_0^{\pi/4}\int_0^{\sin x}(x-y)\,dy\,dx + \int_{\pi/4}^{\pi/2}\int_0^{\cos x}(x-y)\,dy\,dx}_{\text{this way we set up the integral.}}}$$

(b) the answer should be $\pi\left(\frac{7}{8}-\frac{1}{2\sqrt{2}}\right)-\sqrt{2}+\frac{1}{4}$