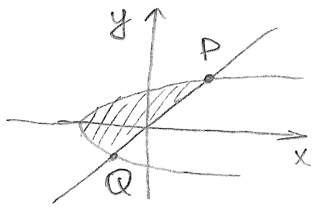
Lecture #12

10/09/2018

* Last time: Double Integrals

Ex1: Evaluate $\iint_D y\,dA$, where D is the region bounded by the line $x=y$ and the parabola $x=y^2-2$.

►



Solve $\underbrace{y^2-2=y}_{}$ to find y-coordinates of points P, Q

$y=-1$ or $y=2$

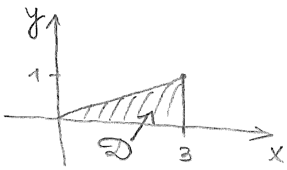
So: $$\iint_D y\,dA = \int_{-1}^{2}\int_{y^2-2}^{y} y\,dx\,dy = \int_{-1}^{2} y(y-y^2+2)\,dy = \left(\frac{y^3}{3}-\frac{y^4}{4}+y^2\right)\Big|_{y=-1}^{y=2} = \boxed{\frac{9}{4}}$$ ∎

Ex2: Evaluate $\int_0^1\int_{3y}^{3} e^{x^2}\,dx\,dy$

►

There is no way to compute the inner integral!

Instead, we shall rewrite it as an iterated integral ending at dydx:

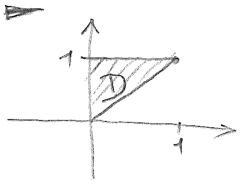


$$\int_0^1\int_{3y}^{3} e^{x^2}\,dx\,dy = \iint_D e^{x^2}\,dA = \int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx =$$

$$= \int_0^3 e^{x^2}\cdot\frac{x}{3}\,dx \xrightarrow[du=2x\,dx]{u=x^2} \int_0^9 e^{u}\cdot\frac{du}{6} = \boxed{\frac{e^9-1}{6}}$$ ∎

Ex3: Evaluate $\iint_D \cos(2y^2)\,dA$, where $D=\{(x,y)\mid 0\le y\le 1,\ 0\le x\le y\}$

►



$$\iint_D \cos(2y^2)\,dA = \int_0^1\int_0^{y} \cos(2y^2)\,dx\,dy = \int_0^1 \cos(2y^2)\cdot y\,dy = \frac{\sin(2y^2)}{4}\Big|_{y=0}^{y=1} = \boxed{\frac{\sin(2)}{4}}$$

Warning: If you write rather as $\int_0^1\int_x^1 \cos(2y^2)\,dy\,dx$ — you will not be able to compute. ∎

Lecture #12 10/09/2018

* Today: Double Integrals via Polar Coordinates (Section 15.3 of textbook)

If we have to integrate $\iint_D f(x,y)\,dA$, where D for example a unit disk $D=\{(x,y)\mid x^2+y^2\le 1\}$, then it is not very convenient to use our previous method as you will get e.g. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$ and the presence of $\sqrt{1-y^2}$ is gonna mess up the computations.

Instead: Use polar coordinates (r,θ)

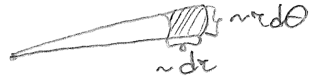
[Recall: $x=r\cos\theta$, $y=r\sin\theta$]

Def: Polar rectangle is a region in $\mathbb{R}^2$ given in polar coordinates by
$$R=\{(r,\theta)\mid \underset{\overset{\vee\!\shortparallel}{0}}{a}\le r\le b,\ \alpha\le\theta\le\beta\}$$
$\beta-\alpha\le 2\pi$

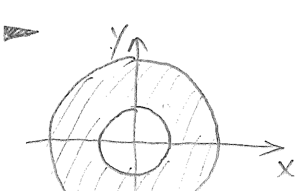
Thm: If $f(\cdot,\cdot)$ is continuous in polar rectangle $R=\{(r,\theta)\mid {a\le r\le b \atop \alpha\le\theta\le\beta}\}$, then
$$\boxed{\iint_R f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\cdot r\cdot dr\,d\theta}$$

Never forget this extra r!

Rmk: The reason for the fact that dA is replaced by $r\,dr\,d\theta$ comes from the following picture:



Ex 4: Evaluate $\iint_R 4(2x^2-y^2)\,dA$, where R is bounded by $x^2+y^2=1$, $x^2+y^2=9$.



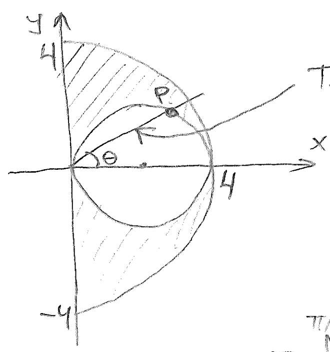
$$\iint_R 4(2x^2-y^2)\,dA=\int_0^{2\pi}\int_1^3 4(2r^2\cos^2\theta-r^2\sin^2\theta)\,r\,dr\,d\theta=$$
$$=\int_0^{2\pi}(160\cos^2\theta-80\sin^2\theta)\,d\theta \overset{\text{double angle}}{=} \int_0^{2\pi}\left(160\cdot\frac{1+\cos(2\theta)}{2}-80\cdot\frac{1-\cos(2\theta)}{2}\right)d\theta$$
$$=\boxed{80\pi}$$

Thm: If $f(\cdot,\cdot)$ is continuous on the polar region of the form $D=\{(r,\theta)\mid \alpha\le\theta\le\beta,\ h_1(\theta)\le r\le h_2(\theta)\}$, then

$$\boxed{\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta}$$

Ex5: Compute $\iint_D y\,dA$, where D is the region bounded by $x^2+y^2=16$ and $x^2+y^2=4x$ in the half-plane $x\ge 0$.

▶ To draw the region, note that $x^2+y^2=4x \Leftrightarrow (x-2)^2+y^2=4$.



To find this distance i.e. r-coordinate for point P, solve $r^2=4r\cos\theta \underset{r>0}{\Longrightarrow} r=4\cos\theta$

So: $\iint_D y\,dA=\int_{-\pi/2}^{\pi/2}\int_{4\cos\theta}^{4} r\sin\theta\cdot r\,dr\,d\theta=$

$=\int_{-\pi/2}^{\pi/2}\sin\theta\left(\frac{81}{3}-\frac{81}{3}\cos^3\theta\right)d\theta$

$\int_{-\pi/2}^{\pi/2}\sin\theta\,d\theta=-\cos\theta\Big|_{\theta=-\pi/2}^{\theta=\pi/2}=0$

$\int_{-\pi/2}^{\pi/2}\sin\theta\cos^3\theta\,d\theta=-\int_{-\pi/2}^{\pi/2}\cos^3(\theta)\,d(\cos\theta)=-\frac{\cos^4\theta}{4}\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}$

$\Rightarrow \iint_D y\,dA=\boxed{0}$

Rmk: The fact that you get ZERO is not surprising as our region is symmetric w.r.t. reflection in x-axis, while our function (y) is skew-symmetric.