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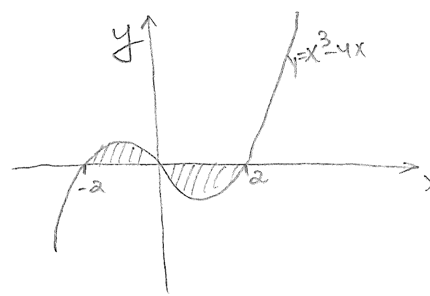
Lecture #13

- Last 2 Lectures: Double integrals (+ polar coordinates)

Ex1: Set up iterated integral for $\iint_D x^4\,dA$, where D is bounded by $y=x^3-4x$ and $y=0$

▶

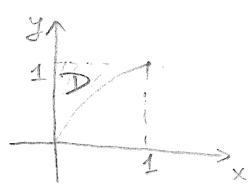


$$\iint_D x^4\,dA = \boxed{\int_{-2}^{0}\int_{0}^{x^3-4x} x^4\,dy\,dx + \int_{0}^{2}\int_{x^3-4x}^{0} x^4\,dy\,dx}$$

(if you compute, you shall get 64/3)

Ex2: Evaluate $\int_0^1\int_{\sqrt{x}}^{1}\sqrt{y^3+1}\,dy\,dx$

▶



$$\int_0^1\int_{\sqrt{x}}^{1}\sqrt{y^3+1}\,dy\,dx = \iint_D \sqrt{y^3+1}\,dA = \int_0^1\int_0^{y^2}\sqrt{y^3+1}\,dx\,dy$$

$$= \int_0^1 y^2\sqrt{y^3+1}\,dy \overset{u=y^3+1}{=\!=\!=} \int_1^2 \sqrt{u}\,\frac{du}{3} = \frac{2}{9}u^{3/2}\Big|_{u=1}^{u=2} = \boxed{\frac{2}{9}(2\sqrt{2}-1)}$$

Ex3: Find the volume of the solid bounded by $z=0$ and $z=4-x^2-y^2$.

▶ $$Vol = \iint_D |4-x^2-y^2|\,dA = \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_{r=0}^{r=2} d\theta = 2\pi\cdot 4 = \boxed{8\pi}$$

* Today: Vector fields (Section 16.1)

Let D be a region in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function F that assigns to each point $(x,y)\in D$ a two-dimensional vector $F(x,y)$

$$F(x,y) = \langle P(x,y),\ Q(x,y)\rangle$$

scalar fields

Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function that assigns to each point $(x,y,z)\in E$ a 3-dimensional vector $F(x,y,z)$

$$F(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$$

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Ex4: Describe the following vector fields by sketching these vector fields at some points and explaining in words (if possible):

(a) $F(x,y) = y\vec{i} - x\vec{j}$

(b) $F(x,y,z) = y\vec{i}$

(c) $F(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$

Recall the notion of a gradient vector field:

$$\boxed{\begin{array}{l} f(x,y) \rightsquigarrow \nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \\ f(x,y,z) \rightsquigarrow \nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \end{array}}$$

Ex5: Find the gradient vector field of:

(a) $f(x,y) = e^x \cdot \sin(2xy)$

(b) $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$

Note: Gradient vector fields are always perpendicular to level curves.

* Line Integrals (Section 16.2)

Let us be given a curve C parametrized as $\vec{r}(t) = \langle x(t), y(t)\rangle$ $(a \leq t \leq b)$

Then: The line integral of $f(x,y)$ along C is defined as

$$(*_1) \quad \boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Remarks: (1) This definition is independent of parametrization of C

(2) If $f(x,y) = 1$, we recover the length f-la $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

(3) Geometrically, if $f(x,y) \geq 0$, then $\int_C f(x,y)\,ds$ equals the area of the "fence" above the curve

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Completely analogously, for a smooth curve C in $\mathbb{R}^3$ given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$, and a continuous function $f(x,y,z)$, define

$$(*_2) \quad \boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt}$$

Note: You may think about the f-las $(*_1)$ & $(*_2)$ as just replacing ds by either $\sqrt{x'(t)^2+y'(t)^2}\,dt$ or $\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$.

Ex6: Evaluate $\int_C (5+xy^2)\,ds$, where C – unit circle.

Parametrize C via $(\cos\theta, \sin\theta)$, $0 \le \theta \le 2\pi$.

$$\int_C (5+xy^2)\,ds = \int_0^{2\pi} (5+\cos\theta\cdot(\sin\theta)^2)\cdot\underbrace{\sqrt{(-\sin\theta)^2+(\cos\theta)^2}}_{1}\,d\theta$$

$$= \int_0^{2\pi} (5+\cos\theta\cdot(\sin\theta)^2)\,d\theta = 10\pi + \int_0^{2\pi} \sin^2(\theta)\,d(\sin\theta) = \boxed{10\pi}$$

Ex7: Evaluate:

(a) $\int_C 3y\,ds$, $C = \{(t^2, 2t) \mid 0 \le t \le 3\}$

(b) $\int_C xe^{2yz}\,ds$, C – line segment from $(0,0,0)$ to $(1,3,2)$.

(a) $\int_C 3y\,ds = \int_0^3 3\cdot 2t\cdot\sqrt{4t^2+4}\,dt = \int_0^3 12t\sqrt{t^2+1}\,dt \overset{u=t^2+1}{=\!=} \int_1^{10} 6\sqrt{u}\,du = \boxed{4(10^{3/2}-1)}$ ✓

(b) C: $(t, 3t, 2t)$, $0 \le t \le 1$.

$$\int_C xe^{2yz}\,ds = \int_0^1 t\cdot e^{12t^2}\cdot\sqrt{1^2+3^2+2^2}\,dt = \sqrt{14}\int_0^1 te^{12t^2}\,dt \overset{u=12t^2}{=\!=} \sqrt{14}\int_0^{12} e^u\,\frac{du}{24} = \frac{\sqrt{14}}{24}(e^{12}-1)$$