

* Last time $\int_C f(x,y) ds$ and $\int_C f(x,y,z) ds$

Last time we learnt the notion of line integral of f along a curve C .

in \mathbb{R}^2 : $\int_C f(x,y) ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ where $C: \langle x(t), y(t) \rangle, a \leq t \leq b$.

in \mathbb{R}^3 $\int_C f(x,y,z) ds = \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$ where $C: \langle x(t), y(t), z(t) \rangle, a \leq t \leq b$.

Remarks: (1) An easy way to remember is that $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$
integrated in the arclength formula

(2) If C is a line segment from $(a, 0, 0)$ to $(b, 0, 0)$, then we recover the definite integrals from high school. $a \leq b$

* Today: Line integrals of f along C with respect to x, y, z .

in \mathbb{R}^2

$$\int_C f(x,y) dx = \int_a^b f(x(t), y(t)) \cdot x'(t) dt$$

$$\int_C f(x,y) dy = \int_a^b f(x(t), y(t)) \cdot y'(t) dt$$

Notational: $\int_C P(x,y) dx + \int_C Q(x,y) dy = \int_C P(x,y) dx + Q(x,y) dy$.

in \mathbb{R}^3 :

$$\int_C f(x,y,z) dx = \int_a^b f(x(t), y(t), z(t)) \cdot x'(t) dt$$

$$\int_C f(x,y,z) dy = \int_a^b f(x(t), y(t), z(t)) \cdot y'(t) dt$$

$$\int_C f(x,y,z) dz = \int_a^b f(x(t), y(t), z(t)) \cdot z'(t) dt$$

Notational: $\int_C P(x,y,z) dx + \int_C Q(x,y,z) dy + \int_C R(x,y,z) dz = \int_C P(x,y,z) dx + Q(x,y,z) dy + R(x,y,z) dz$

! To distinguish $\int_C f ds$, they are sometimes called line integrals wrt arc length

Ex1: Evaluate $\int_C x dx + y dy$ along the following curves:

(a) C - line segment from $(-1, 0)$ to $(1, 0)$

(b) C - half of the unit circle going clockwise from $(-1, 0)$ to $(1, 0)$

(c) C - part of the parabola $y = x^2 - 1$, $-1 \leq x \leq 1$ (going from $(-1, 0)$ to $(1, 0)$)

(a) $C = \{(t, 0) \mid -1 \leq t \leq 1\} \Rightarrow \int_C x dx + y dy = \int_{-1}^1 t dt = \boxed{0}$

(b) $C = \{(\cos t, \sin t) \mid t \text{ ranges from } \pi \text{ to } 0\}$

$\Rightarrow \int_C x dx + y dy = \int_{\pi}^0 (\cos t (-\sin t) + \sin t \cdot \cos t) dt = \boxed{0}$

(c) $C = \{(t, t^2 - 1) \mid -1 \leq t \leq 1\} \Rightarrow \int_C x dx + y dy = \int_{-1}^1 t dt + (t^2 - 1) \cdot 2t dt = \int_{-1}^1 (2t^3 - t) dt = \boxed{0}$

Note: We got the same answer for different 3 curves.

But this is not always the case!

Ex2: Evaluate $\int_C e^x dx + xy dy$ for the following curves:

(a) C - part of parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$

(b) C - line segment from $(0, 0)$ to $(2, 4)$

(c) C consists of line segment from $(0, 0)$ to $(1, 1)$ followed up by a line segment from $(1, 1)$ to $(2, 4)$.

(a) $C = \{(t, t^2) \mid 0 \leq t \leq 2\} \Rightarrow \int_C e^x dx + xy dy = \int_0^2 (e^t + t \cdot t^2 \cdot 2t) dt = (e^t + \frac{2}{5} t^5) \Big|_{t=0}^{t=2} = e^2 - 1 + \frac{64}{5}$

(b) $C = \{(2t, 4t) \mid 0 \leq t \leq 1\} \Rightarrow \int_C e^x dx + xy dy = \int_0^1 (e^{2t} \cdot 2 dt + 2t \cdot 4t \cdot 4 dt) = (e^{2t} + \frac{32}{3} t^3) \Big|_{t=0}^{t=1} = e^2 - 1 + \frac{32}{3}$

(c) $C = C_1 \cup C_2$, $C_1 = \{(t, t) \mid 0 \leq t \leq 1\}$, $C_2 = \{(1+t, 1+3t) \mid 0 \leq t \leq 1\}$.

$\int_{C_1} e^x dx + xy dy = \int_0^1 (e^t + t^2) dt = (e^t + \frac{t^3}{3}) \Big|_{t=0}^{t=1} = e - 1 + \frac{1}{3}$

$\int_{C_2} e^x dx + xy dy = \int_0^1 e^{1+t} dt + (1+t)(1+3t) \cdot 3 dt = \int_0^1 [e^{1+t} + 3(3t^2 + 4t + 1)] dt = (e^{1+t} + 3(t^3 + 2t^2 + t)) \Big|_{t=0}^{t=1} = e^2 - e + 3 \cdot 4$

$\Rightarrow \int_C e^x dx + xy dy = \int_{C_1} \dots + \int_{C_2} \dots = e^2 - 1 + \frac{1}{3} + 12$

Remark: In Ex 2 we got different integrals while varying curve.

As we will see later on, the reason why in Ex 1 the answer was independent of the curve is b/c. $x dx + y dy$ is a gradient vector field of $\frac{1}{2}(x^2 + y^2)$.

Remark: Note that $\int_C f(x,y) ds = \int_{-C} f(x,y) ds$, $\int_C f(x,y) dx = -\int_{-C} f(x,y) dx$,

$$\int_C f(x,y) dy = -\int_{-C} f(x,y) dy$$

where $-C$ denotes the same curve C but passed in the opposite direction.

Ex 3: Evaluate $\int_C xye^{yz} dy$, $C: x=t, y=t^2, z=t^3, 0 \leq t \leq 1$

$$\int_0^1 t \cdot t^2 \cdot e^{t^3} \cdot 2t dt = \int_0^1 2t^4 e^{t^3} dt = \frac{2}{5} e^{t^3} \Big|_{t=0}^{t=1} = \frac{2}{5}(e-1)$$

* Line Integrals of Vector Fields

Def: Let \vec{F} be a continuous vector field on a smooth curve C given by a vector function $\vec{r}(t)$, $a \leq t \leq b$.

Then the line integral of \vec{F} along C is:

$$\int_C \vec{F} d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

↑ also equals $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{T} ds$, \vec{T} - unit tangent vector at the given point

Ex 4: Evaluate $\int_C F dr$, where $F(x,y,z) = x\vec{i} + y^2\vec{j} + z^3\vec{k}$, while C is given by $x=t, y=t^2, z=t^3, 0 \leq t \leq 1$.

$$\begin{aligned} \int_C F dr &= \int_0^1 (t\vec{i} + t^4\vec{j} + t^9\vec{k}) \cdot (t\vec{i} + 2t\vec{j} + 3t^2\vec{k}) dt \\ &= \int_0^1 (t + 2t^5 + 3t^9) dt = \left(\frac{t^2}{2} + \frac{t^6}{3} + \frac{t^{10}}{10} \right) \Big|_{t=0}^{t=1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} \end{aligned}$$

Note that if $F = P\vec{i} + Q\vec{j} + R\vec{k}$, $\vec{z}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\int_C F d\vec{z} = \int_a^b [P(x(t), y(t), z(t)) \cdot x'(t) + Q(x(t), y(t), z(t)) y'(t) + R(x(t), y(t), z(t)) z'(t)] dt$$

$$\int_C F d\vec{z} = \int_a^b P dx + Q dy + R dz$$

\Rightarrow recover the same kind of line integral that we discussed in the beginning of today's class.

Ex 5: Evaluate the line integral $\int_C F d\vec{z}$ where

$$F(x, y) = e^x \vec{i} - \sin(y) \vec{j}, \quad C: \vec{z}(t) = t^2 \vec{i} + t^3 \vec{j}, \quad 0 \leq t \leq 1$$

$$F(x, y) = \langle e^x, -\sin y \rangle \quad \vec{z}(t) = \langle t^2, t^3 \rangle \Rightarrow F(\vec{z}(t)) = \langle e^{t^2}, -\sin(t^3) \rangle$$

$$\vec{z}'(t) = \langle 2t, 3t^2 \rangle$$

$$\text{So: } \int_C F d\vec{z} = \int_0^1 (2t e^{t^2} - 3t^2 \sin(t^3)) dt = \left[e^{t^2} + \cos(t^3) \right] \Big|_{t=0}^{t=1} = e - 1 + \cos(1) - 1$$

$$= \boxed{e + \cos(1) - 2}$$

Ex 6: Evaluate $\int_C F d\vec{z}$, where $F(x, y) = xy^2 \vec{i} - x^3 \vec{j}$

$$C: \vec{z}(t) = t^3 \vec{i} + t^4 \vec{j}, \quad 0 \leq t \leq 1.$$

$$\int_C F d\vec{z} = \int_0^1 \langle t^4, -t^9 \rangle \cdot \langle 3t^2, 4t^3 \rangle dt = \int_0^1 (3t^{13} - 4t^{12}) dt = \left(\frac{3}{14} t^{14} - \frac{4}{13} t^{13} \right) \Big|_{t=0}^{t=1}$$

$$= \boxed{\frac{3}{14} - \frac{4}{13}}$$

Ex 7: Evaluate $\int_C F d\vec{z}$, where $F(x, y, z) = \sin x \cdot \vec{i} + \cos y \cdot \vec{j} + xz \cdot \vec{k}$

$$C: \vec{z}(t) = t^3 \vec{i} - t^2 \vec{j} + t \vec{k}, \quad 0 \leq t \leq 1$$

$$\int_C F d\vec{z} = \int_0^1 \langle \sin(t^3), \cos(-t^2), t^4 \rangle \cdot \langle 3t^2, -2t, 1 \rangle dt$$

$$= \left(-\cos(t^3) + \sin(-t^2) + \frac{1}{5} t^5 \right) \Big|_{t=0}^{t=1} = 1 - \cos(1) + \sin(-1) + \frac{1}{5} = \boxed{\frac{6}{5} - \cos(1) - \sin(1)}$$