* <u>Last time</u>

   Last time we discussed line integrals of vector fields $\int_C \vec{F}\,d\vec{r}$.
   and we saw that it is the same as $\int_C P\,dx + Q\,dy$ or $\int_C P\,dx + Q\,dy + R\,dz$,
   where $\vec{F} = <P,Q>$ (if we are in the plane) or $\vec{F} = <P,Q,R>$ (if we are in the space)

   And what we saw in the first two Exercises last time is that usually
   $\int_C \vec{F}\,d\vec{r}$ depends not only on the end & start points of C, but also on C
   itself, while for $\vec{F} = <x,y>$ we obtained the same answer for 3 different
                                    curves with the same end/start points!

<u>Question</u>: What is special about $<x,y>$?
<u>Answer</u>: It is a gradient vector field as $<x,y> = \nabla\left(\frac{1}{2}(x^2+y^2)\right)$

<u>Def</u>: A vector field $\vec{F}$ is called <u>conservative</u> if
   (1) $\vec{F} = \nabla f$ for some f ← such f is called a <u>potential function</u>
   (2) the components of $\vec{F}$ are defined and have continuous partials everywhere

<u>Ex1</u>: Is the vector field $\vec{F} = <-y, x>$ conservative?

   Actually it is not, but let's assume at the moment we found
   $f(x,y)$ s.t. $\vec{F} = \nabla f \Longleftrightarrow \begin{cases} f_x = -y \\ f_y = x \end{cases} \Rightarrow (-y)_y = (x)_x \Rightarrow$ Contradiction!

   <u>Recall</u>: $(f_x)_y = (f_y)_x$ $\underset{-1}{"} \quad \underset{1}{"}$

<u>So</u>: There is no such $f \Rightarrow \vec{F}$ is not conservative
The above reasoning implies:

<u>Thm1</u>: Let $\vec{F} = <P,Q>$ be a vector field s.t. $P_y \neq Q_x$, then
         $\vec{F}$ is <u>not</u> conservative.

Ex2: Is the vector field $\vec{F} = \langle x+y, \sin(y) + 2x + 3\rangle$ conservative?

In this case $P = x+y$, $Q = \sin(y) + 2x + 3$

$$P_y = 1 \qquad\qquad Q_x = 2.$$

As $P_y(x,y) \neq Q_x(x,y) \implies \vec{F}$ is <u>not</u> conservative

Actually the converse is also true, namely we have:

Thm2: If $\vec{F} = \langle P, Q\rangle$ is a vector field whose components are defined and have continuous partials everywhere, and $P_y = Q_x$, then $\vec{F}$ is conservative.

Note: This theorem doesn't tell how to find potentials!
We shall illustrate the approach by the following problem.

Ex3: (a) Is the vector field $\vec{F} = \langle x+y, x+y^2\rangle$ conservative?
    (b) If yes, find a potential of $\vec{F}$.

(a) $x+y$ & $x+y^2$ clearly have continuous partials $\Big\}\implies \vec{F}$ is conservative.
$(x+y)_y = 1 = (x+y^2)_x$

(b) By part (a), we know that there exists $f(x,y)$ s.t. $\nabla f = \vec{F}$, i.e.
$f_x = x+y$, $f_y = x+y^2$.

Step 1
Integrating $x+y$ w.r.t. x, we see that $f(x,y) = \frac{1}{2}x^2 + xy + g(y)$ ← for some function of y

Step 2
$f_y(x,y) = 0 + x + g'(y) \implies g'(y) = y^2 \implies g(y) = \frac{y^3}{3} + C_0$
(underbrace: $x+y^2$)   ↑ constant.

So: For any constant C_0: $\boxed{f(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{3}y^3 + C_0}$ is a potential of $\vec{F}$.

Ex 4: Is the vector field $\vec{F} = \langle x+z, y-z, x+y \rangle$ conservative?

Assume yes, i.e. there is some function $f(x,y,z)$ s.t. $\nabla f = \vec{F}$, i.e.

$$f_x = \underbrace{x+z}_{P}, \quad f_y = \underbrace{y-z}_{Q}, \quad f_z = \underbrace{x+y}_{R}.$$

Similarly to our reasoning in Ex1, recall that

$$(f_x)_y = (f_y)_x, \qquad (f_x)_z = (f_z)_x, \qquad (f_y)_z = (f_z)_y.$$

$$\Downarrow \qquad\qquad\qquad \Downarrow \qquad\qquad\qquad \Downarrow$$

$$P_y = Q_x \qquad\qquad P_z = R_x \qquad\qquad Q_z = R_y$$

In our case

$$(x+z)_y = 0 = (y-z)_x \qquad \checkmark$$

$$(x+z)_z = 1 = (x+y)_x \qquad \checkmark$$

$$(y-z)_z = -1 \neq 1 = (x+y)_y \qquad -$$

So: there is no such $f(x,y,z) \Rightarrow \vec{F}$ is <u>not</u> conservative

---

Theorem 3: Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field s.t.

$$P_y \neq Q_x \quad \text{or} \quad P_z = R_x \quad \text{or} \quad Q_z \neq R_y,$$

then $\vec{F}$ is <u>not</u> conservative.

The converse turns out to be also true:

Theorem 4: If $\vec{F} = \langle P, Q, R \rangle$ is a vector field whose components are defined and have continuous partials everywhere, and

$$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y,$$

then $\vec{F}$ <u>is</u> conservative

Once again, let me point out that this theorem does not specify how to find potential.

To find it we shall follow the strategy of Ex 3(b), but we will need 3 steps now

Ex5: (a) Is the vector field $\vec{F} = \langle y\cos(xy)+z,\ x\cos(xy)+2yz,\ x+y^2 \rangle$ conservative?

(b) If yes, find a potential.

▶ (a) Just need to verify three equalities

$$(y\cos(xy)+z)_y = \cos(xy) - xy\sin(xy) = (x\cos(xy)+2yz)_x$$

$$(y\cos(xy)+z)_z = 1 = (x+y^2)_x$$

$$(x\cos(xy)+2yz)_z = 2y = (x+y^2)_y$$

As also partial derivatives of components are continuous everywhere, we see that $\vec{F}$ is conservative.

(b) Want: Find $f(x,y,z)$ s.t. $f_x = y\cos(xy)+z,\ f_y = x\cos(xy)+2yz,\ f_z = x+y^2$.

Step 1: $f_x = y\cos(xy)+z$

Treating y, z as constants and integrating w.r.t. x, we get

$$\boxed{f(x,y,z) = \sin(xy) + xz + g(y,z)}$$

Step 2: $f_y = \underset{\substack{\| \\ x\cos(xy)+0+g_y}}{x\cos(xy)+2yz} \Big\} \Rightarrow g_y = 2yz \Rightarrow \boxed{g(y,z) = y^2 z + h(z)}$

$$\Rightarrow \boxed{f(x,y,z) = xz + \sin(xy) + y^2 z + h(z)}$$

Step 3: $\underset{\substack{\| \\ x+y^2+h'(z)}}{f_z = x+y^2} \Big\} \Rightarrow h'(z) = 0 \Rightarrow h(z) = c_0 \leftarrow$ constant.

Hence: Potentials are: $\boxed{f(x,y,z) = xz + y^2 z + \sin(xy) + c_0}$

Ex6: Same two questions for the vector field
$$\vec{F} = \langle e^{x+y} + 2x+y^2,\ e^{x+y}+2xy+y^2 \rangle.$$

Recall: Fundamental theorem of Calculus from high-school says

$$\int_a^b G'(x)\,dx = G(b) - G(a)$$

The following generalization is known under the name "Fundamental Theorem of Line Integrals" (FTLI)

Thm 5: If $\vec{F}$ is a conservative vector field with $\vec{F} = \nabla f$, and C is a curve from point A to point B, then

$$\int_C \vec{F}\,d\vec{r} = f(B) - f(A)$$

Ex 7: Let $\vec{F} = \langle x^2, y \rangle$ and C be a line segment from $(0,0)$ to $(1,1)$. Evaluate $\int_C \vec{F}\,d\vec{r}$.

▸ $\vec{F}$ is conservative and $\vec{F} = \nabla f\underbrace{(\frac{1}{3}x^3 + \frac{1}{2}y^2)}_{f(x,y)}$

Hence: FTLI $\Rightarrow \int_C \vec{F}\,d\vec{r} = f(1,1) - f(0,0) = \frac{1}{3} + \frac{1}{2} = \boxed{\frac{5}{6}}$ ▨

Note: In this simple case, you could also compute $\int_C \vec{F}\,d\vec{r}$ in a straightforward way.

Ex 8: If $\vec{F} = \langle x+y, x+y^2 \rangle$ and C is a counterclockwise portion of $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$ followed by the line segment from $(-1,0)$ to $(1,1)$, evaluate $\int_C \vec{F}\,d\vec{r}$.

▸ According to Ex 3(b) $f(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{3}y^3$ is a potential of $\vec{F}$.

Hence: FTLI $\Rightarrow \int_C \vec{F}\,d\vec{r} = f(-1,1) - f(1,0) = \frac{1}{2} - 1 + \frac{1}{3} - \frac{1}{2} = \boxed{-\frac{2}{3}}$ ▨

Summary of today's discussion:

Given a line integral $\int_C \vec{F}\, d\vec{z}$:

(1) Verify if $\vec{F}$ is conservative or not (through the mixed partials test)

(2) If Yes, then either find a potential and apply FTLI

$\underline{\underline{or}}$ replace C by a simpler path C' b/w the same two points and compute directly

$$\int_{C'} \vec{F}\, d\vec{z}.$$