* Last time

Last time we learned the notion of conservative vector fields and the criteria to verify whether a given vector field is conservative or not. However, if the criteria gives a positive answer, we need to find the corresponding potential by iterative integration w.r.t x, y, z.

Finally, we also learned the "FTLI", which allows to find the integrals of conservative vector fields by:

- either finding a potential
- or replacing the curve b/w the start/end-points.

Ex1: Let $\vec{F} = \langle 2xy+z, x^2, x\rangle$ and C is curve given by $\vec{r}(t) = \langle \sin(t^2), \cos(t^2), \cos(t^2)\rangle$ $0 \le t \le \sqrt{\pi}$.

Evaluate $\int_C \vec{F} d\vec{r}$.

- In this problem, you could compute the line integral in a straight-forward way (use $u=t^2$ substitution to evaluate $\int$) given more time.
- Let's however, compute this integral via FTLI.

Step 1

$(2xy+z)_y = 2x = (x^2)_x$, $(2xy+z)_z = 1 = (x)_x$, $(x^2)_z = 0 = (x)_y \Rightarrow \vec{F}$ - conservative

Step 2

Find potential to be $f(x,y,z) = x^2y + xz + C_0$ ← constant

Step 3

Apply FTLI: $\int_C \vec{F} d\vec{r} = f(B) - f(A)$, where $B = \vec{r}(\sqrt{\pi}) = \langle 0, -1, -1\rangle$

$A = \vec{r}(0) = \langle 0, 1, 1\rangle$

$\Rightarrow \int_C \vec{F} d\vec{r} = \boxed{0}$

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Ex 2: Let $\vec{F} = \langle 2x+3y, y^2+3x+e^{\sin(y)} \rangle$, C – top half of the unit circle oriented counterclockwise. Compute $\int_C \vec{F}\,d\vec{r}$.

- First, we check if $\vec{F}$ is conservative: $(2x+3y)_y = 3 = (y^2+3x+e^{\sin(y)})_x$
  $$\Downarrow$$
  $\vec{F}$ – conservative.

- Now, we can try to find potential f of $\vec{F}$, i.e. a function of two variables s.t. $f_x = 2x+3y$, $f_y = y^2+3x+e^{\sin(y)}$.

  From the first equality we find $f(x,y) = x^2+3xy+g(y)$ ← a function of y.
  But from the second equality, we get
  $$y^2+3x+e^{\sin(y)} = f_y = 3x + g'(y) \Rightarrow g'(y) = \underbrace{y^2+e^{\sin(y)}}$$
  ↑ there is no closed formula for antiderivative of $e^{\sin(y)}$!

So: Unlike all the previous examples, we know that $\vec{F}$ is conservative, but we cannot find an explicit potential.

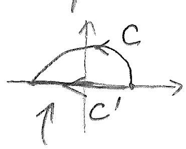
Route #1

Let $g(y)$ be an antiderivative of $y^2+e^{\sin(y)}$ (it does exist for sure!).
Then $f(x,y) = x^2+3xy+g(y)$ – potential of $\vec{F}$
$\Downarrow$ FTLI

$$\int_C \vec{F}\,d\vec{r} = f(-1,0) - f(1,0) = ((-1)^2+0+g(0)) - (1^2+0+g(0)) = \boxed{0} \left(\begin{array}{c}\text{so } g(0) \text{ got}\\ \text{cancelled}\end{array}\right)$$

Route #2

Replace the curve and compute explicitly:



$$\int_C \vec{F}\,d\vec{r} = \int_{C'} \vec{F}\,d\vec{r} = \int_1^{-1} \langle 2t, 3t\rangle \cdot \langle 1,0\rangle\,dt = \int_1^{-1} 2t\,dt = \boxed{0}$$

C': $(t,0)$, t goes from 1 to −1

Note that the FTLI guarantees that $\int_C \vec{F}\,d\vec{r} = 0$ if C – closed path, $\vec{F}$ – conservative vector field.

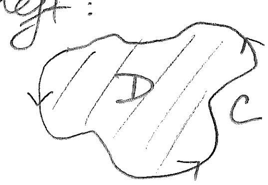
* Today: Green's Theorem

Today we shall learn another very important tool, which allows to compute $\int_C \vec{F}\,d\vec{r}$ over closed paths by reducing them to double integrals.

Theorem (Green's Theorem): Let $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ be a vector field and C be a closed, positively oriented curve enclosing a region D, and assume that P, Q have continuous partials on D. Then:

$$\boxed{\int_C \vec{F}\,d\vec{r} = \iint_D (Q_x - P_y)\,dA}$$

‖ C is positively oriented if walking along C in this direction, D is always on the left:



Note: If $\vec{F}$ – conservative $\Rightarrow Q_x - P_y = 0 \Rightarrow \int_C \vec{F}\,d\vec{r} = 0$ as already observed above.

Ex3: Let C be a unit circle oriented counterclockwise. Compute $\int_C (x^2+y)\,dx + (y + e^{\sin(y)} + x^2)\,dy$.

Using Green's Thm: $\int_C (x^2+y)\,dx + (y+e^{\sin(y)}+x^2)\,dy = \int_C \vec{F}\,d\vec{r} =$, where $\vec{F} = \langle x^2+y,\, y+e^{\sin(y)}+x^2\rangle$

$$= \iint_{D\text{-unit disk}} (2x-1)\,dA = \int_0^{2\pi}\int_0^1 (2r\cos\theta - 1)\cdot r\,dr\,d\theta = \boxed{-\pi}$$

∎

Note: If C – closed path, negatively oriented, then we reduce to the positive oriented setup above via $\int_C \vec{F}\,d\vec{r} = -\int_{-C} \vec{F}\,d\vec{r}$.

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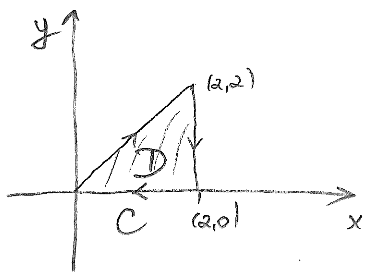
Ex 4: Let $\vec{F} = \langle \sin(x), x^2y^3 \rangle$ and C be the triangle with vertices $(0,0)$, $(2,0)$, $(2,2)$, oriented clockwise. Compute $\int_C \vec{F}\,d\vec{r}$.



▶ As this orientation is negative, applying Green's Thm we get

$$\int_C \vec{F}\,d\vec{r} = -\int_{-C} \vec{F}\,d\vec{r} = -\iint_D \left((x^2y^3)_x - (\sin x)_y\right) dA =$$

$$= -\iint_D 2xy^3\,dA = -\int_0^2\int_0^x 2xy^3\,dy\,dx = -\int_0^2 \left(\frac{xy^4}{2}\Big|_{y=0}^{y=x}\right) dx =$$

$$= -\int_0^2 \frac{x^5}{2}\,dx = -\frac{x^6}{12}\Big|_{x=0}^{x=2} = -\frac{64}{12} = \boxed{-\frac{16}{3}}$$

■

Finally, let us investigate a problem where one wants to apply the Green's Theorem, but the path C is not closed.

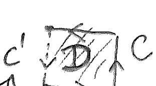
Hint: • Close up the path as follows:

• apply Green's Thm to evaluate $\int_{C\cup C'} \vec{F}\,d\vec{r}$

• evaluate in a straightforward way $\int_{C'} \vec{F}\,d\vec{r}$.

Ex 5: Let $\vec{F} = \langle x^3 + y, y^2 \rangle$ and C be a path from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ along straight line segments. Evaluate $\int_C \vec{F}\,d\vec{r}$.

▶ One can compute this in a straightforward way, but let us do it much faster using Green's Thm.



$(0,t)$, t goes from 1 to 0

$$\int_{C\cup C'} \vec{F}\,d\vec{r} = \iint_D [(y^2)_x - (x^3+y)_y]\,dA = -\iint_D 1\cdot dA = -\text{Area}(D) = -1$$

$$\| $$

$$\int_C \vec{F}\,d\vec{r} + \int_{C'} \vec{F}\,d\vec{r}$$

$$\int_{C'} \vec{F}\,d\vec{r} = \int_1^0 \langle t, t^2\rangle \cdot \langle 0,1\rangle\,dt = \frac{t^3}{3}\Big|_{t=1}^{t=0} = -\frac{1}{3}$$

$$\Rightarrow \int_C \vec{F}\,d\vec{r} = \boxed{-\frac{2}{3}}$$

■

! It is a good exercise to compute $\int_C \vec{F}\,d\vec{r}$ in Ex 5 in a straightforward way (over each line segment separately)