* Last Week: FTLI and Green's Theorem.

Ex1: Let $\vec{F} = \langle e^{x^2}, e^{y^2} \rangle$ and C be a curve given by $\vec{r}(t) = \langle t\sqrt{2}\cos(\frac{7\pi}{4}t), t\sqrt{2}\sin(\frac{7\pi}{4}t) \rangle$, $0 \le t \le 1$

Evaluate $\int_C \vec{F}\,d\vec{r}$.

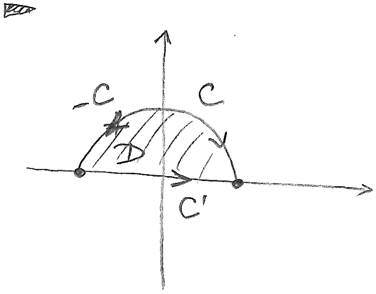
▶ We start by noticing immediately that $(e^{x^2})_y = 0 = (e^{y^2})_x \Rightarrow \vec{F}$ - conservative.
However, it is impossible to find explicitly an antiderivative of e^{x^2}
$\Rightarrow$ we cannot find an explicit formula for a potential of $\vec{F}$.

Instead: replace the path.
Starting point is $A = \vec{r}(0) = \langle 0,0 \rangle$, the end point is $B = \vec{r}(1) = \langle 1,-1 \rangle$

Let C' be the line segment from $(0,0)$ to $(1,-1)$
parametrized by $\vec{r}(t) = (t,-t)$, $0 \le t \le 1$.

$\vec{F}$ - conservative $\Rightarrow \int_C \vec{F}\,d\vec{r} = \int_{C'} \vec{F}\,d\vec{r} = \int_0^1 \langle e^{t^2}, e^{t^2} \rangle \cdot \langle 1,-1 \rangle\,dt = \int_0^1 0\,dt = \boxed{0}$ ■

Ex2: Let $\vec{F} = \langle y^2x + x^2, x^2y + x - y^{y\sin(y)} \rangle$, C - top half of $x^2+y^2=1$ oriented clockwise.
Evaluate $\int_C \vec{F}\,d\vec{r}$.

▶ 

As always let $-C$ denote the same curve, but oriented in the opposite way.
Also let C' denote the line segment from $(-1,0)$ to $(1,0)$.

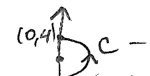
Then: by Green's Theorem:

$$\int_{(-C)\cup C'} \vec{F}\,d\vec{r} = \iint_D \left[(x^2y + x - y^{y\sin y})_x - (y^2x + x^2)_y\right] dA = \iint_D dA = \text{Area}(D) = \frac{\pi}{2}$$

$$\int_{C'} \vec{F}\,d\vec{r} = \int_{-1}^{1} \langle t^2, t \rangle \cdot \langle 1,0 \rangle\,dt = \int_{-1}^{1} t^2\,dt = \frac{t^3}{3}\Big|_{t=-1}^{t=1} = \frac{2}{3}$$

parametrize C' via $(t,0)$, $-1 \le t \le 1$

$$\Rightarrow \int_{-C} \vec{F}\,d\vec{r} = \frac{\pi}{2} - \frac{2}{3} \Rightarrow \int_C \vec{F}\,d\vec{r} = -\int_{-C} \vec{F}\,d\vec{r} = \boxed{\frac{2}{3} - \frac{\pi}{2}}$$ ■

Exercise (to do at home): Compute $\int_C e^{x^2}dx + \cos y\,dy$ where 

C - half circle.

* Hand out the print-outs summarizing the strategy to compute line integrals.
* Sketch of the proof of FTLI (p.1088 of textbook)

$$\int_C \nabla f \, d\vec{r} = \int_a^b \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t} \right) dt \overset{\text{Chain Rule}}{=} \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt \overset{\text{Fundamental Theorem of Calculus}}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$

* Recall that given a continuous vector field $\vec{F}$ with domain D, we say $\int_C \vec{F} d\vec{r}$ is <u>independent of path</u> if $\int_{C_1} \vec{F} d\vec{r} = \int_{C_2} \vec{F} d\vec{r}$ for any two paths C_1, C_2 in D with the same initial and terminal points.

According to what we discussed two classes ago:

<u>Theorem</u>: $\int_C \vec{F} d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} d\vec{r}$ for every <u>closed</u> path C in D
meaning that initial and terminal points coincide

<u>Theorem</u>: (a) If $\vec{F}$ is conservative, then FTLI implies that $\int_C \vec{F} d\vec{r}$ is independent of path.

(b) If $\vec{F}$ is such that $\int_C \vec{F} d\vec{r}$ is independent of path and D is simply-connected (i.e. does not contain any hole), then $\vec{F}$ is conservative.

! These should be used for homework problem 16.3.30.

<u>Ex3</u>: Show that the line integral $\int_C x\,dx + yz\,dy + e^y z^2\,dz$ is <u>not</u> independent of path.

– Just note that $(yz)_z = y \neq e^y z^2 = (e^y z^2)_y$

* It is common to write $\oint_C \vec{F} d\vec{r}$ if C is closed and positively oriented.
* When applying Green's Theorem:
  - close the path C (by adding C') if it is not closed from the beginning
  - change orientation if C is negatively oriented (use $\int_C \dots = -\int_{-C} \dots$)

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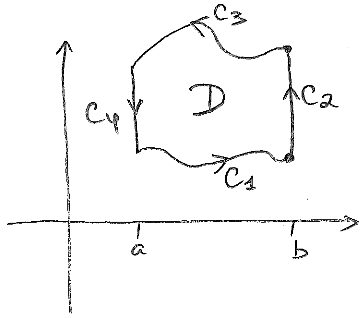
* Sketch of the proof of Green's Thm in the particular case when D is given by $D=\{(x,y) \mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$ and $Q(x,y)=0$ (see p. 1097 of textbook)



$$\iint_D P_y\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)}\frac{\partial P}{\partial y}(x,y)\,dy\,dx \underset{\text{Fund. Theorem of calculus.}}{=} \int_a^b\left[P(x,g_2(x))-P(x,g_1(x))\right]dx$$

$$\int_{C_1}P(x,y)dx=\int_a^b P(x,g_1(x))dx,\quad \int_{C_3}P(x,y)dx=-\int_a^b P(x,g_2(x))dx$$

$$\int_{C_2}P(x,y)dx=0=\int_{C_4}P(x,y)dx$$

$$\Rightarrow \int_C P(x,y)dx=\int_{C_1}P(x,y)dx+\int_{C_2}P(x,y)dx+\int_{C_3}P(x,y)dx+\int_{C_4}P(x,y)dx$$

$$=\int_a^b\left[P(x,g_1(x))-P(x,g_2(x))\right]dx=-\iint_D\frac{\partial P}{\partial y}\,dA$$

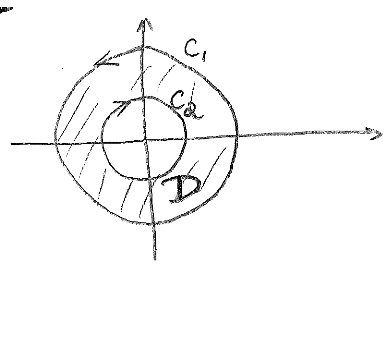
Exercise (to do at home): Evaluate the line integral $\oint_C (x^2+y)dx-y^3dy$ in two ways: (C ← the unit circle)

(a) directly

(b) using Green's Theorem. (you should get $\boxed{-\pi}$)

Let us do one more exercise before we switch to a new topic:

Ex 4: Evaluate $\oint_C (e^{x^2}+2y)dx+(\sin(y^{10})+8x)dy$, where C is the boundary of the region between circles $x^2+y^2=1$ and $x^2+y^2=4$.



Given a region D, its boundary is always viewed as being positively oriented.

In our case, the boundary C consists of big circle C_1 oriented counterclockwise and a small circle C_2 oriented clockwise.

Green's Theorem: $\oint_C (e^{x^2}+2y)dx+(\sin(y^{10})+8x)dy=\iint_D\left[(\sin(y^{10})+8x)_x-(e^{x^2}+2y)_y\right]dA$

$$=\iint_D 6\,dA=6\cdot \text{Area}(D)=6\cdot(\pi\cdot 2^2-\pi\cdot 1^2)=\boxed{18\pi}$$

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* Note that Green's Theorem can be used other way around: reducing double integral to a line integral.

For example:

$$\boxed{\text{Area}(D) = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx, \quad \text{where } C \text{ is the boundary of } D.}$$

Ex 5: Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

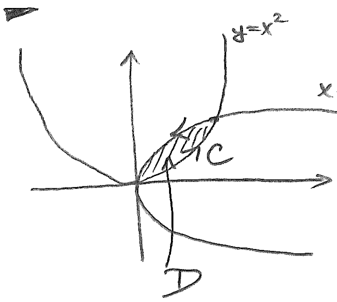
The boundary C may be parametrized via $\vec{r}(t)=\langle a\cos t, b\sin t\rangle$, $0\le t\le 2\pi$.

So: $\text{Area}(D) = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}[(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)]\,dt =$

$= \frac{1}{2}\int_0^{2\pi} ab\underbrace{(\cos^2 t + \sin^2 t)}_{1}\,dt = \boxed{\pi ab}$

Ok, one more exercise on Green:

Ex 6: Evaluate $\oint_C (3y+e^{2\sqrt{x}})\,dx + (x+\sin(y^3))\,dy$, where C – the boundary of the region bounded by $y=x^2$ and $x=y^2$.



$\oint_C (3y+e^{2\sqrt{x}})\,dx + (x+\sin(y^3))\,dy =$

$= \iint_D (1-3)\,dA = -2\int_0^1\int_{x^2}^{\sqrt{x}} 1\,dy\,dx = -2\int_0^1 (x^{1/2}-x^2)\,dx =$

$= -2\left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_{x=0}^{x=1} = -2\cdot\frac{1}{3} = -\frac{2}{3}$

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* Today: "Curl" and "Divergence"

Given a vector field $\vec{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ s.t. partial derivatives of P, Q, R exist, the curl of $\vec{F}$ is defined as follows:

$$\boxed{\text{curl } \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle}$$

vector differential operator

To simplify memorizing this formula, we may use $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ to write

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \text{curl}(\vec{F})$$

So: $\boxed{\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F}}$

Given a vector field $\vec{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ s.t. partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ exist, the divergence of $\vec{F}$ is a function of 3 variables defined by:

$$\boxed{\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

As above, we can simplify this by:

$$\boxed{\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F}}$$

Ex 7: Find the curl and divergence of $\vec{F} = xye^z\vec{i} + \sin(yz)\vec{j} + xze^y\vec{k}$.

$\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F} = ye^z + z\cos(yz) + xe^y$

$\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = (xze^y - y\cos(yz))\vec{i} - (ze^y - xye^z)\vec{j} - xe^z\vec{k}$ ☐

Theorem: (1) $\text{curl}(\nabla f) = 0$ for any function $f(x,y,z)$ that has continuous second order partial derivatives.

(2) If $\vec{F}$ is a vector field defined on all $\mathbb{R}^3$, whose components have continuous partial derivatives and $\text{curl}(\vec{F}) = \vec{0}$, then $\vec{F}$ is conservative, i.e. $\vec{F} = \nabla f$ for some f

(1) follows from $\text{curl}(\nabla f) = \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle$.

Ex8: Determine whether the following vector field is conservative or not. If it is conservative, find a potential.

(a) $\vec{F} = \langle e^y \cos x, xye^z, z\sin y \rangle$

(b) $\vec{F} = \langle 1 + e^y + ze^x, xe^y, e^x \rangle$.

(a) The coefficient of $\vec{i}$ in $\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F}$ is $z\cos y - xye^z \neq 0 \Rightarrow \vec{F}$ is not conservative!

(b) $$\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1+e^y+ze^x & xe^y & e^x \end{vmatrix} = \vec{0} \Rightarrow \vec{F} \text{ is conservative.}$$

Potential: $f(x,y,z) = ze^x + xe^y + x + c_0$ ← constant

Ex 9: Compute divergence of the curl($\vec{F}$) from Ex7.

$$\text{div}(\text{curl}(\vec{F})) = ze^y - ze^y + xe^z - xe^z = \boxed{0}$$

This answer is not accidental as we have:

> **Theorem:** If $\vec{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second-order partial derivatives, then
> $$\text{div}(\text{curl}\,\vec{F}) = 0.$$

$$\text{div}(\text{curl}\,\vec{F}) = \text{div}\left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \right)$$
$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0$$

The opposite is also true, but there are two many choices of $\vec{G}$ s.t. $\text{curl}(\vec{G}) = \vec{F}$ if $\text{div}(\vec{F}) = 0$ (Indeed add any gradient vector field). So let us just practice on an example.

Ex 10: Determine whether or not a vector field $\vec{F}$ is a curl of some $\vec{G}$. If it is, find any $\vec{G}$ s.t. $\vec{F} = \text{curl}(\vec{G})$.

(a) $\vec{F} = \langle x\sin z, y^2 + xz, x + \cos z \rangle$

(b) $\vec{F} = \langle -y, z, y \rangle$

(a) If $\vec{F} = \text{curl}(\vec{G})$, then $\text{div}(\vec{F})$ should be ZERO.

However: $\text{div}(\vec{F}) = \sin z + 2y - \sin z = 2y \neq 0 \Rightarrow \vec{F}$ is not a curl.

(b) $\text{div}\,\vec{F} = 0 + 0 + 0 = 0 \Rightarrow \vec{F}$ is a curl, i.e. there is $\vec{G}$: $\vec{F} = \text{curl}(\vec{G})$.

Let us try to find such $\vec{G} = \langle P, Q, R \rangle$.

* First, we start from $R_y - Q_z = -y$

  Choose R any way: the simplest choice is $R = 0 \Rightarrow -Q_z = -y$

  $\Rightarrow Q = yz + g(x,y)$.

* Analogously: $P_z - R_x = z \overset{R=0}{\Rightarrow} P_z = z \Rightarrow P = \frac{z^2}{2} + h(x,y)$

* Finally: $\underbrace{Q_x - P_y}_{g_x - h_y} = y \Rightarrow g_x - h_y = y$.

  Again, there are many choices, e.g. $h(x,y) = 0$, $g(x,y) = xy$.

Thus: $\vec{F} = \text{curl}\left(\langle \frac{z^2}{2}, yz + xy, 0 \rangle\right)$