Lecture # 18 11/01/2018

* Organizational

↳ Please fill out the midterm surveys by Friday's night

↳ Check out Ximera for explanations and practice problems on line integrals.

* Last time: Curl and Divergence.

Thm: (1) $\text{curl}(\nabla f) = 0$ for any function $f(x,y,z)$ that has continuous second order partial derivatives.

(2) If $\vec{F}$ is a vector field defined on all $\mathbb{R}^3$ whose components have continuous derivatives and $\text{curl}(\vec{F}) = \vec{0}$, then $\vec{F} = \nabla f$ for some f

! Discuss Ex 8-10 and Theorem from pages 6-7 of Lecture #17.

! Hand out a printout with an anti-curl example.

Attached to these notes.

Ex 1: Prove $\text{div}(f \cdot \vec{F}) = f \cdot \text{div}(\vec{F}) + \vec{F} \cdot \nabla f$

$\vec{F} = \langle P, Q, R \rangle \Rightarrow f\vec{F} = \langle fP, fQ, fR \rangle$

$\text{div}(f\vec{F}) = \frac{\partial(fP)}{\partial x} + \frac{\partial(fQ)}{\partial y} + \frac{\partial(fR)}{\partial z} = (f_x \cdot P + f \cdot P_x) + (f_y \cdot Q + f \cdot Q_y) + (f_z \cdot R + f \cdot R_z)$

$= f(P_x + Q_y + R_z) + (Pf_x + Qf_y + Rf_z) = f \,\text{div}\, \vec{F} + \vec{F} \cdot \nabla f$ ∎

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*Today: Parametric surfaces (Section 16.6 of your textbook)

Analogously to curves, we can describe a surface by a vector function $\vec{r}(u,v)$ of two parameters u, v:

$$\boxed{\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}}$$ – vector-valued function defined on $D \subseteq \mathbb{R}^2_{uv}$.

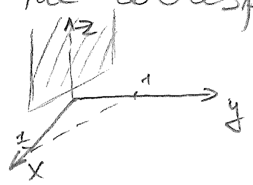
Def: The set of all points $(x,y,z) \in \mathbb{R}^3$ such that $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ for some $(u,v) \in D$ is called a parametric surface

Ex2: Identify and sketch the following surfaces with vector equation:

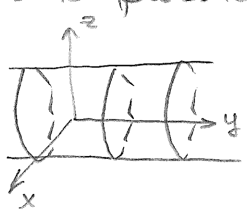
(a) $\vec{r}(u,v) = (1-2u)\,\vec{i} + 2u\cdot\vec{j} + e^{v^2}\cdot\vec{k}$ with $D = \{(u,v) \mid u \in \mathbb{R}, v \in \mathbb{R}_{\geq 0}\}$

(b) $\vec{r}(u,v) = 5\sin u \cdot \vec{i} - 10v \cdot \vec{j} + 5\cos u \cdot \vec{k}$

(a) The corresponding surface is the part of the plane $x+y=1$ with $z \geq 1$:



(b) The corresponding surface is the cylinder whose base is a circle of radius 5 in the xz-plane centered at the origin and whose axis is parallel to y-axis:



If we keep u constant by putting $u = u_0$, then $\vec{r}(u_0, v)$ defines a curve C_1 on S. Likewise, if we keep v constant by putting $v = v_0$, then $\vec{r}(u, v_0)$ defines a curve C_2 on S. These curves are called grid curves.

Ex3: Describe grid curves for two surfaces from Ex2.

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Ex 4: Find a parametric representation of the plane passing through the point $P_0(x_0, y_0, z_0)$ and containing two non-parallel vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

$$\begin{cases} x = x_0 + a_1 \cdot u + b_1 v \\ y = y_0 + a_2 \cdot u + b_2 v \\ z = z_0 + a_3 \cdot u + b_3 v \end{cases}$$ ◻

Ex 5: Find a parametric equation of the cylinder $x^2+y^2=9$, $-2 \le z \le 5$.

$$\begin{cases} x = 3\cos u \\ y = 3\sin u \\ z = v \end{cases} \quad \text{with} \quad \begin{matrix} 0 \le u \le 2\pi \\ -2 \le v \le 5 \end{matrix}$$ ◻

Ex6: Find a parametric equation of the graph of a function $f(x,y)$ on $\mathbb{R}^2$.

$$\begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$$ ◻

Ex 7: Find a parametric equation of the top half (i.e. $z \ge 0$) of the cone $z^2 = 16(x^2+y^2)$

One way: $x = u, y = v, z = 4\sqrt{u^2+v^2}$ $\quad (u,v) \in \mathbb{R}^2$

Second way: $x = u\cos v, y = u\sin v, z = 4u$ $\quad u \ge 0, 0 \le v \le 2\pi$ ◻

Ex 8 (Surface of revolution): Let S be obtained by rotating the curve $y = f(x)$, $a \le x \le b$ (assume $f(x) \ge 0$) about the x-axis. Find a parametric equation of S.

$x = u$, $y = f(u)\cos v$, $z = f(u)\sin v$, $\quad a \le u \le b$, $0 \le v \le 2\pi$ ◻

Ex8: Determine whether the following vector field is conservative or not. If it is conservative, find a potential.

(a) $\vec{F} = \langle e^y \cos x,\ xye^z,\ z\sin y \rangle$

(b) $\vec{F} = \langle 1 + e^y + ze^x,\ xe^y,\ e^x \rangle$.

(a) The coefficient of $\vec{i}$ in $\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F}$ is $z\cos y - xye^z \neq 0 \Rightarrow \vec{F}$ is not conservative!

(b) $$\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1+e^y+ze^x & xe^y & e^x \end{vmatrix} = \vec{0} \Rightarrow \vec{F} \text{ is conservative.}$$

Potential: $f(x,y,z) = ze^x + xe^y + x + c_0$ ← constant

Ex 9: Compute divergence of the $\operatorname{curl}(\vec{F})$ from Ex7.

$\operatorname{div}(\operatorname{curl}(\vec{F})) = ze^y - ze^y + xe^z - xe^z = \boxed{0}$

This answer is not accidental as we have:

> Theorem: If $\vec{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second-order partial derivatives, then
> $$\operatorname{div}(\operatorname{curl} \vec{F}) = 0.$$

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \operatorname{div}\left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right)$$
$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0$$

The opposite is also true, but there are two many choices of $\vec{G}$ s.t. $\operatorname{curl}(\vec{G}) = \vec{F}$ if $\operatorname{div}(\vec{F}) = 0$ (Indeed add any gradient vector field). So let us just practice on an example.

Ex 10: Determine whether or not a vector field $\vec{F}$ is a curl of some $\vec{G}$. If it is, find any $\vec{G}$ s.t. $\vec{F} = \text{curl}(\vec{G})$.

(a) $\vec{F} = \langle x\sin z, y^2 + xz, x + \cos z \rangle$

(b) $\vec{F} = \langle -y, z, y \rangle$

(a) If $\vec{F} = \text{curl}(\vec{G})$, then $\text{div}(\vec{F})$ should be ZERO.

However: $\text{div}(\vec{F}) = \sin z + 2y - \sin z = 2y \neq 0 \Rightarrow \vec{F}$ is not a curl.

(b) $\text{div}\,\vec{F} = 0 + 0 + 0 = 0 \Rightarrow \vec{F}$ is a curl, i.e. there is $\vec{G}$: $\vec{F} = \text{curl}(\vec{G})$.

Let us try to find such $\vec{G} = \langle P, Q, R \rangle$.

* First, we start from $R_y - Q_z = -y$

Choose R any way: the simplest choice is $R = 0 \Rightarrow -Q_z = -y$

$\Rightarrow Q = yz + g(x,y)$.

* Analogously: $P_z - R_x = z \overset{R=0}{\Rightarrow} P_z = z \Rightarrow P = \frac{z^2}{2} + h(x,y)$

* Finally: $\underbrace{Q_x - P_y}_{g_x - h_y} = y \Rightarrow g_x - h_y = y$.

Again, there are many choices, e.g. $h(x,y) = 0$, $g(x,y) = xy$.

Thus: $\vec{F} = \text{curl}\left(\langle \frac{z^2}{2}, yz + xy, 0 \rangle\right)$

Theorem: If $\vec{F}$ is a vector field whose components have continuous partials everywhere, and $\text{div}(\vec{F}) = 0$, then $\vec{F} = \text{curl}(\vec{G})$ for some $\vec{G}$.