Lecture #19 11/06/2018

* Last time : Parametric Surfaces

Ex1 : Let Γ be the graph of $y=x^2$, $0\le x\le 2$, in the xy-plane.

(a) Parametrize the surface S_1 obtained by rotating Γ around x-axis.

(b) Parametrize the surface S_2 obtained by rotating Γ around y-axis.

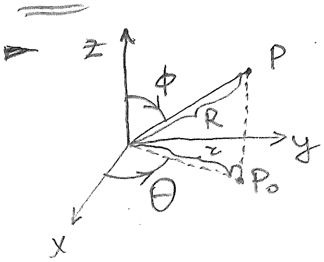
(a) S_1 : $\vec{r}(u,v) = \langle u, u^2\cos v, u^2 \sin v\rangle$, where $0\le u\le 2$, $0\le v\le 2\pi$.

(b) S_1 : $\vec{r}(u,v) = \langle u\cos v, u^2, u\sin v\rangle$, where $0\le u\le 2$, $0\le v\le 2\pi$

or some of you could also write

$\vec{r}(u,v) = \langle \sqrt{u}\cdot\cos v, u, \sqrt{u}\cdot \sin v\rangle$, where $0\le u\le 4$, $0\le v\le 2\pi$.

Ex2 : Find a parametric equation of the sphere $x^2+y^2+z^2=R^2$.



- $z = R\cos\phi$
- $x = r\cos\theta$, $y = r\sin\theta$

But $r = R\sin\phi$

$$\Rightarrow \boxed{\begin{cases} x = R\sin\phi\cos\theta \\ y = R\sin\phi\sin\theta \\ z = R\cos\phi \end{cases} ,\quad \begin{array}{l} 0\le\theta\le 2\pi \\ 0\le\phi\le\pi \\ \text{not } 2\pi! \end{array}}$$

! Hand out matching game.

* Today : Grid Curves, Tangent Planes, Surface Area.

Grid Curves

If we keep u constant by putting $u=u_0$, then $\vec{r}(u_0,v)$ defines a curve C_1 on S.
If we keep v constant by putting $v=v_0$, then $\vec{r}(u,v_0)$ defines a curve C_2 on S.
These curves C_1 and C_2 are called grid curves.

Ex3 : (a) Describe grid curves for both surfaces of Ex1.

(b) Describe grid curves for the parametrization of a sphere in Ex2.

(c) Describe grid curves for the cylinder $x=R\cos u$, $y=R\sin u$, $z=v$.

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Tangent Planes

Goal: Find the tangent plane to the parametric surface S traced out by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ at the point $P_0 = \vec{r}(u_0, v_0)$

As we know, the tangent plane contains tangent vectors at P_0 to various curves on S passing through P_0. In particular, we have tangent vectors to the grid curves C_1 and C_2 passing through P_0.

C_1: traced out by $\vec{r}(u_0,v) \Rightarrow$ tangent vector is $\frac{\partial}{\partial v}\vec{r}(u_0,v)$, which equals $\boxed{\vec{r}_v = \langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0)\rangle}$

C_2: traced out by $\vec{r}(u,v_0) \Rightarrow$ tangent vector is $\frac{\partial}{\partial u}\vec{r}(u,v_0)$, which equals $\boxed{\vec{r}_u = \langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0)\rangle}$

|| S is smooth if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ for any P_0 on S

Moral of the above discussion: The tangent plane to S at P_0 is determined by:

(1) P_0 belongs to this plane

(2) $\vec{n} = \vec{r}_u \times \vec{r}_v$ is a normal vector.

Ex 4: (a) Describe the surface S parametrized by
$$\vec{r}(u,v) = u^2\,\vec{i} + 2u\cdot\sin v\cdot\vec{j} + u\cos v\cdot\vec{k}.$$

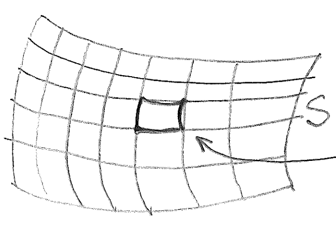
(b) Find the tangent plane to S at the point P_0 given by $\vec{r}(1,0)$

(a) Note that $(2u\sin v)^2 + 4\cdot(u\cos v)^2 = 4u^2 \Rightarrow$ any point $(x,y,z)\in S$ satisfies $4x = y^2+4z^2$. The converse is also clear. Thus, S is defined by this equation $4x = y^2+4z^2 \Rightarrow S$ - elliptic paraboloid.

(b) $\vec{r}_u = \langle 2,0,1\rangle$, $\vec{r}_v = \langle 0,2,0\rangle \Rightarrow \vec{n} = \vec{r}_u \times \vec{r}_v = \langle -2,0,4\rangle$
$P_0 = \vec{r}(1,0) = (1,0,1)$ $\Big\} \Rightarrow$

$\Rightarrow$ Tangent Plane: $-2(x-1) + 4(z-1) = 0 \iff \boxed{x - 2z = -1}$

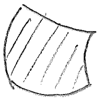
Surface Area



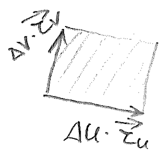
Looking at the grid curves on S, we see that S is divided into small "patches". Every such patch can be approximated by a parallelogram with sides determined by vectors

$\Delta u \cdot \vec{r}_u$ and $\Delta v \cdot \vec{r}_v$.

Patch



is approximated by



But as we discussed in the first week of the class, the area of a parallelogram equals the magnitude of the cross-product

$$|(\Delta u \vec{r}_u) \times (\Delta v \cdot \vec{r}_v)| = \Delta u \cdot \Delta v \cdot |\vec{r}_u \times \vec{r}_v|.$$

This explains the following definition:

Def (Surface Area): If a smooth parametric surface S is given by $\vec{r}(u,v)$, $(u,v) \in D$, and S is covered just once as (u,v) ranges through D, then the surface area of S is

$$\boxed{A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA.}$$

Now we can easily define a surface integral of a function.

Def (Surface Integral of a function): Under the same conditions as in the previous definition, the surface integral of $f(x,y,z)$ over the surface S is:

$$\boxed{\iint_S f(x,y,z) \, dS = \iint_D f(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v| \, dA}$$