* Last time

Last time we discussed how to parametrize a sphere and also introduced the formula for the surface area and a surface integral of a function over a surface.

Recall: If S is parametrized by $\vec{r}(u,v)$, $(u,v)\in D$ ← domain, then:

$$\boxed{A(S)=\iint_D |\vec{r}_u\times\vec{r}_v|\,dA}$$ – surface integral

while a surface integral of $f(x,y,z)$ over S is

$$\boxed{\iint_S f\,dS=\iint_D f(\vec{r}(u,v))\,|\vec{r}_u\times\vec{r}_v|\,dA}$$

Ex1: (a) Find the surface area of a sphere $S: x^2+y^2+z^2=R^2$.

(b) Find the surface integral $\iint_S x^2\,dS$, where S – as in (a).

Step 0: Parametrize S as we did last time:

$$\boxed{\vec{r}(\phi,\theta)=\langle R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi\rangle\ ;\ D=\left\{(\phi,\theta)\ \middle|\ \begin{matrix}0\le\phi\le\pi\\0\le\theta\le2\pi\end{matrix}\right\}}$$

Step 1: Evaluate $\vec{r}_\phi\times\vec{r}_\theta$.

$$\left.\begin{aligned}\vec{r}_\phi&=\langle R\cos\phi\cos\theta,\ R\cos\phi\sin\theta,\ -R\sin\phi\rangle\\ \vec{r}_\theta&=\langle -R\sin\phi\sin\theta,\ R\sin\phi\cos\theta,\ 0\rangle\end{aligned}\right\}\Rightarrow \boxed{\vec{r}_\phi\times\vec{r}_\theta=\begin{aligned}&R^2\sin^2\phi\cos\theta\cdot\vec{i}\ +\\&R^2\sin^2\phi\sin\theta\cdot\vec{j}\ +\\&R^2\sin\phi\cos\phi\cdot\vec{k}\end{aligned}}$$

Step 2: Evaluate $|\vec{r}_\phi\times\vec{r}_\theta|$

$$|\vec{r}_\phi\times\vec{r}_\theta|=\sqrt{R^4\sin^4\phi\cos^2\theta+R^4\sin^4\phi\sin^2\theta+R^4\sin^2\phi\cos^2\phi}\overset{\text{as } \sin^2\theta+\cos^2\theta=1}{=}\sqrt{R^4\sin^4\phi+R^4\sin^2\phi\cos^2\phi}$$

$$=\sqrt{R^4\sin^2\phi(\sin^2\phi+\cos^2\phi)}=\sqrt{R^4\sin^2\phi}=|R^2\sin\phi|\underset{\text{as } 0\le\phi\le\pi}{=}R^2\sin\phi$$

So: $\boxed{|\vec{r}_\phi\times\vec{r}_\theta|=R^2\sin\phi}$

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(Continuation of Ex 1)

(a) $A(S)=\iint_D R^2\sin\phi\, dA=\int_0^{\pi}\underbrace{\int_0^{2\pi} R^2\sin\phi\, d\theta}_{2\pi R^2\sin\phi}\, d\phi=-2\pi R^2\cos\phi\Big|_{\phi=0}^{\phi=\pi}=\boxed{4\pi R^2}$

(b) In part (b), we reduce almost to same integral, but the function we integrate is $\underbrace{(R^2\sin^2\phi\cos^2\theta)}_{\text{this is } x^2!}\cdot\underbrace{(R^2\sin\phi)}_{\text{this is our } |\vec{r}_\phi\times\vec{r}_\theta|}$

$$\iint_S x^2 dS=\iint_D R^2\sin^2\phi\cos^2\theta\cdot R^2\sin\phi\, dA=R^4\int_0^{\pi}\int_0^{2\pi}\sin^3\phi\cdot\cos^2\theta\, d\theta\, d\phi=$$

$$=R^4\cdot\int_0^{\pi}\sin^3\phi\, d\phi\cdot\int_0^{2\pi}\cos^2\theta\, d\theta$$

But: $\int_0^{2\pi}\cos^2\theta\, d\theta=\int_0^{2\pi}\frac{1+\cos(2\theta)}{2}\, d\theta=\pi$

$$\int_0^{\pi}\sin^3\phi\, d\phi=\int_0^{\pi}\sin^2\phi\cdot\sin\phi\, d\phi=\int_0^{\pi}(1-\cos^2\phi)\, d(-\cos\phi)\overset{u=-\cos\phi}{=\!=}\int_{-1}^{1}(1-u^2)\, du=\left(u-\frac{u^3}{3}\right)\Big|_{u=-1}^{u=1}=\frac{4}{3}$$

Thus: $\boxed{\iint_S x^2 dS=\frac{4}{3}\pi R^4}$

Remark: One way to view the surface integral is as follows. Consider the sheet of aluminium foil that has shape of a surface S and the density at point (x,y,z) is $\rho(x,y,z)$, then the total mass of the sheet is $m=\iint_S \rho(x,y,z)\, dS$, while the coordinates of the center of mass are:

$$\left(\frac{1}{m}\iint_S x\rho(x,y,z)\, dS,\ \frac{1}{m}\iint_S y\rho(x,y,z)\, dS,\ \frac{1}{m}\iint_S z\rho(x,y,z)\, dS\right)$$

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* Special case of interest: graphs of functions

A very frequent example of a surface is a graph of a function in 2 variables.
If S is a graph of $f(x,y)$, $(x,y)\in D$, then one natural parametrization is

$$\boxed{\vec{r}(u,v) = u\cdot\vec{i} + v\cdot\vec{j} + f(u,v)\,\vec{k}\quad,\ (u,v)\in D}$$

Then: $\left.\begin{array}{l}\vec{r}_u = \langle 1, 0, f_u\rangle \\ \vec{r}_v = \langle 0, 1, f_v\rangle\end{array}\right\} \Rightarrow \vec{r}_u\times\vec{r}_v = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & f_u \\ 0 & 1 & f_v\end{vmatrix} = -f_u\cdot\vec{i} - f_v\cdot\vec{j} + \vec{k}$

So: $\boxed{\vec{r}_u\times\vec{r}_v = -f_u\cdot\vec{i} - f_v\cdot\vec{j} + \vec{k} \Rightarrow |\vec{r}_u\times\vec{r}_v| = \sqrt{1+f_u^2+f_v^2}}$

Thus:
$$\boxed{\begin{array}{l} A(S) = \displaystyle\iint_D \sqrt{1+(f_u)^2+(f_v)^2}\,dA \\ \displaystyle\iint_S g(x,y,z)\,dS = \iint_D g(u,v,f(u,v))\cdot\sqrt{1+(f_u)^2+(f_v)^2}\,dA\end{array}}$$

Rmk: Similar formulas apply when S is realized as a graph of a function in x,z or y,z, i.e. $y=f(x,z)$ or $x=f(y,z)$, respectively.

Ex2: Find the area of the part S of the paraboloid $y=x^2+z^2$ that lies within the cylinder $x^2+z^2=16$.

▸ Viewing S as a graph of a function $y=x^2+z^2$ with $(x,z)\in D=\{(u,v)\mid u^2+v^2\le 16\}$ the above formula implies

$$A(S) = \iint_{D:\,u^2+v^2\le16} \sqrt{1+(2u)^2+(2v)^2}\,dA = \iint_{D-\text{disk } u^2+v^2\le 16} \sqrt{1+4u^2+4v^2}\,dA \quad \underline{\text{Polar coordinates}}$$

$$= \underbrace{\int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\cdot r\,dr\,d\theta}_{u=1+4r^2 \Rightarrow du=8r\,dr \Rightarrow \int_0^4\sqrt{1+4r^2}\,r\,dr = \int_1^{65}\sqrt{u}\,\frac{du}{8} = \left(\frac{1}{12}u^{3/2}\right)\Big|_1^{u=65} = \frac{1}{12}(65^{3/2}-1)} = \frac{2\pi}{12}(65^{3/2}-1) = \boxed{\frac{\pi}{6}(65^{3/2}-1)}$$

■

Remark: Note that while switching from dA to $dr\,d\theta$ we always get extra r. However, if we use polar coordinates in the very beginning of parametrization the surface, we do not need this r.

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Ex3: Recalculate $A(S)$ in Ex2 using another parametrization
$$\vec{r}(u,v) = \langle u\cos v,\ u^2,\ u\sin v\rangle, \quad 0\le u\le 4,\ 0\le v\le 2\pi.$$

$$\left.\begin{array}{l}\vec{r}_u = \langle \cos v, 2u, \sin v\rangle \\ \vec{r}_v = \langle -u\sin v, 0, u\cos v\rangle\end{array}\right\} \Rightarrow \vec{r}_u\times\vec{r}_v = \langle 2u^2\cos v, -u, 2u^2\sin v\rangle$$
$$\Downarrow$$
$$|\vec{r}_u\times\vec{r}_v| = \sqrt{4u^4\cos^2 v + u^2 + 4u^4\sin^2 v} = \sqrt{u^2+4u^4} = u\cdot\sqrt{1+4u^2}.$$

So: $$A(S) = \underbrace{\iint_{\substack{0\le u\le 4\\ 0\le v\le 2\pi}} u\cdot\sqrt{1+4u^2}\,dA}_{}$$
this is exactly the same integral we got in Ex2 after using polar coord.
$\Rightarrow$ we got the same answer $\boxed{\frac{\pi}{6}(65^{3/2}-1)}$

But: What I tried to emphasize in the previous Remark, the factor u before $\sqrt{1+4u^2}$ is already there! ∎

* Today: Flux (or surface integrals of vector fields).
While $\iint_S f\,dS$ was reminiscent of line integrals $\int_C f\,ds$, we want to discuss an analogue of line integrals $\int_C \vec{F}\,d\vec{r}$.
However, even to give the definition, we need to restrict our attention only to a certain class of surfaces, called oriented. Note that at any point $(x,y,z)\in S$ there are two unit normal vectors (i.e. orthogonal to tangent plane)

Def: If it is possible to choose a unit normal vector $\vec{n}$ at every point $(x,y,z)\in S$ so that $\vec{n}$ varies continuously over S, then S is called an Oriented surface and the given choice of $\vec{n}$ provides S with an orientation

Remark: Not every surface is oriented, but every oriented surface has exactly two orientations.

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Def: If $\vec{F}$ is a continuous vector field defined on an oriented surface S with an orientation defined by the unit vector $\vec{n}$ (at each point), then the surface integral of $\vec{F}$ over S (a.k.a. the flux of $\vec{F}$ across S) is

$$\boxed{\iint_S \vec{F}\, dS = \iint_S \underbrace{\vec{F}\cdot\vec{n}}\, dS}$$

vector field

dot-product, hence, a function on S, which we know how to integrate from last lecture.

Main ingredient: find $\vec{n}$!

There are two basic examples:

Example 1: S is a given as a graph of $f(x,y)$, $(x,y)\in D$.

As discussed above, the natural way to parametrize S is by
$\vec{r}(u,v) = u\cdot\vec{i} + v\vec{j} + f(u,v)\vec{k}$, $(u,v)\in D$.

As we found $\vec{r}_u\times\vec{r}_v = \langle -f_u, -f_v, 1\rangle$ – this is a normal, but not a unit vector.

Hence, the unit normal vectors are $\pm\dfrac{-f_u\cdot\vec{i} - f_v\vec{j} + \vec{k}}{\sqrt{1+f_u^2+f_v^2}}$.

Upward Orientation refers to the choice $\boxed{\vec{n} = \dfrac{-f_x\cdot\vec{i} - f_y\vec{j} + \vec{k}}{\sqrt{1+(f_x)^2+(f_y)^2}}}$

Example 2: Smooth parametric surface given by $\vec{r}(u,v)$, $(u,v)\in D$.

Then $\vec{r}_u\times\vec{r}_v$ is a nonzero normal vector, and one of the two choices of the unit normal vector is

$$\boxed{\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}}$$

Def: For a closed surface (i.e. boundary of a solid E), the positive orientation is the one where the normal vector points outside of E, while the inward-pointing normals give the negative orientation.

Back to the Example 2, if we use $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$, then we see

$$\iint_S \vec{F}\,d\vec{S} = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cdot \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA$$

$$\Rightarrow \boxed{\iint_S \vec{F}\,d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA}$$

Ex 4: Find the flux of $\vec{F}(x,y,z) = \langle 3z, 3y, 3x \rangle$ across the sphere S given by $x^2+y^2+z^2=R^2$.

Choose the same parametrization as in Ex 1:

$$\vec{r}(\phi, \theta) = \langle R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi \rangle$$

As we computed in Ex 1: $\boxed{\vec{r}_\phi \times \vec{r}_\theta = \langle R^2\sin^2\phi\cos\theta,\ R^2\sin^2\phi\sin\theta,\ R^2\sin\phi\cos\phi \rangle}$

while $\boxed{\vec{F}(\vec{r}(\phi,\theta)) = \langle 3R\cos\phi,\ 3R\sin\phi\sin\theta,\ 3R\sin\phi\cos\theta \rangle}$

$$\Rightarrow \vec{F}(\vec{r}(\phi,\theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) = 3R^3\left(\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta\right)$$
$$= 3R^3\left(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta\right)$$

So: $\iint_S \vec{F}\,d\vec{S} = \int_0^\pi \int_0^{2\pi} 3R^3\left(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta\right) d\theta\, d\phi$

Note: $\int_0^{2\pi} \cos\theta\, d\theta = 0$, $\int_0^{2\pi} \sin^2\theta\, d\theta = \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\, d\theta = \pi$

$$\Rightarrow \iint_S \vec{F}\,d\vec{S} = 3\pi R^3 \cdot \underbrace{\int_0^\pi \sin^3\phi\, d\phi}_{= \frac{4}{3} \text{ - see Ex 1(b)}} = \boxed{4\pi R^3}$$