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- Last time:

Last time we learnt the notion of the flux of a vector field $\vec{F}$ across the surface S (i.e. the surface integral of $\vec{F}$ over S).

Definition:
$$\boxed{\iint_S \vec{F}\,dS = \iint_S \vec{F}\cdot\vec{n}\,dS} \quad (1)$$

$\vec{n}$ - a choice of a unit normal vector at every point of S, which changes continuously (determined by an orientation of S).

Practical Formula: If we parametrized S via $\underbrace{\vec{r}(u,v)}_{\text{covers } S \text{ without overlaps}}$, $(u,v)\in D$, then

$$\boxed{\iint_S \vec{F}\,dS = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\pm\, \vec{r}_u\times\vec{r}_v)\,dA} \quad (2)$$

you have to determine the sign based on the orientation.

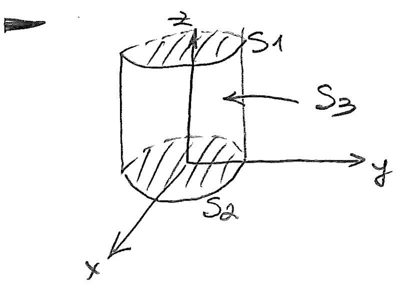
Conventions: For a closed surface S (i.e. S is the boundary of the solid E), the positive orientation is the one where the normal vector points outside of E, while the negative orientation is the one where the normal vector points inwards E.

Thus, the general strategy to compute flux $\iint_S \vec{F}\,dS$ is:

* Split S into several parts, so that you can parametrize each of them
* Compute $\vec{r}_u\times\vec{r}_v$ and decide if you must take $+$ or $-$ in (2)
* Evaluate the dot-product $\vec{F}(\vec{r}(u,v))\cdot(\pm\,\vec{r}_u\times\vec{r}_v)$
* Integrate.

However, in some simple cases you may simplify this general algorithm.

Ex 1: Find the flux of the vector field $\vec{F}(x,y,z) = \langle x,y,z \rangle$ over a cylinder given by $x^2+y^2=9$, $0 \le z \le 5$ together with its top and bottom. The orientation is chosen to be positive.



We split the entire surface S into 3 parts:

S_1 – the disk $x^2+y^2 \le 9$, $z=5$ on top

S_2 – the disk $x^2+y^2 \le 9$, $z=0$ on the bottom

S_3 – side part given by $x^2+y^2=9$, $0 \le z \le 5$.

Clearly: $\iint_S \vec{F}\,dS = \iint_{S_1} \vec{F}\,dS + \iint_{S_2} \vec{F}\,dS + \iint_{S_3} \vec{F}\,dS$ and we need to compute each of these 3 fluxes.

Flux across S_1

1st way: Parametrize S_1 via $\vec{r}(u,v) = \langle u\cos v, u\sin v, 5 \rangle$ $\quad 0 \le u \le 3$, $0 \le v \le 2\pi$

Then $\left.\begin{array}{l} \vec{r}_u = \langle \cos v, \sin v, 0 \rangle \\ \vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle \end{array}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = \vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}\cdot(u\cos^2 v + u\sin^2 v)$

$= u\cdot\vec{k}$ and from picture it's clear $u\vec{k}$ looks outside!

Hence: $\iint_{S_1} \vec{F}\,dS = \int_0^{2\pi}\int_0^3 \langle u\cos v, u\sin v, 5 \rangle \cdot \langle 0,0,u \rangle\,du\,dv = \int_0^{2\pi}\int_0^3 5u\,du\,dv = \boxed{45\pi}$

2nd way: Let us provide an alternative way to compute $\iint_{S_1} \vec{F}\,dS$.

Just from the picture it is clear that $\vec{n} = \vec{k}$ on $S_1 \Rightarrow \vec{F}\cdot\vec{n} = 5$ on S_1

$\Rightarrow \iint_{S_1} \vec{F}\,dS = \iint_{S_1} 5\,dS = 5\cdot \underbrace{A(S_1)}_{\substack{\text{surface Area} \\ \text{of } S_1 = \text{disk of radius } 3}} = 5\cdot\pi\cdot 3^2 = 45\pi$ ← Got the same answer.

Flux across S_2

1st way: Parametrize S_1 via $\vec{r}(u,v) = \langle u\cos v, u\sin v, 0 \rangle$ $\quad 0 \le u \le 3$, $0 \le v \le 2\pi$

As above, we get $\vec{r}_u \times \vec{r}_v = u\cdot\vec{k}$, but looking at the picture we actually see that $u\vec{k}$ points inwards the solid $\Rightarrow$ need to take $-u\cdot\vec{k}$.

$\Rightarrow \iint_{S_2} \vec{F}\,dS = \int_0^{2\pi}\int_0^3 \underbrace{\langle u\cos v, u\sin v, 0 \rangle \cdot \langle 0,0,-u \rangle}_{0}\,du\,dv = \boxed{0}$

2nd Way: Looking at picture $\vec{n} = -\vec{k}$ on $S_2 \Rightarrow \vec{F}\cdot\vec{n} = 0$ on $S_2 \Rightarrow \iint_{S_2} \vec{F}\,dS = \boxed{0}$

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Flux across S_3

1st way: Parametrize S_3 via $\vec{r}(u,v) = \langle 3\cos v, 3\sin v, u\rangle$ $\quad 0 \le u \le 5,\ 0 \le v \le 2\pi$.

$$\left.\begin{array}{l}\vec{r}_u = \langle 0,0,1\rangle \\ \vec{r}_v = \langle -3\sin v, 3\cos v, 0\rangle\end{array}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -3\sin v & 3\cos v & 0 \end{vmatrix} = -3\cos v \cdot \vec{i} - 3\sin v \cdot \vec{j}$$
$$= \langle -3\cos v, -3\sin v, 0\rangle$$

Now we have to decide whether we pick $\vec{r}_u \times \vec{r}_v$ or $-\vec{r}_u \times \vec{r}_v$. We need a vector which points outwards. It suffices to check at any point on S_3. For example when $u=0, v=0$, we get $\vec{r}_u \times \vec{r}_v = \langle -3,0,0\rangle$, while looking back at the picture we see that this points inwards.

So: We need to take $-\vec{r}_u \times \vec{r}_v = \langle 3\cos v, 3\sin v, 0\rangle$

Thus: $$\iint_{S_3} \vec{F}\,d\vec{S} = \int_0^{2\pi}\int_0^5 \underbrace{\langle 3\cos v, 3\sin v, u\rangle \cdot \langle 3\cos v, 3\sin v, 0\rangle}_{9}\, du\, dv = \boxed{90\pi}$$

2nd Way: Looking at the picture, it is clear that $\vec{n}$ is always parallel to xy-plane and is explicitly given by $\vec{n} = \langle \frac{x}{3}, \frac{y}{3}, 0\rangle$ at the point $(x,y,z) \Rightarrow \vec{F}\cdot\vec{n} = \frac{x^2+y^2}{3} = 3$ on S_3

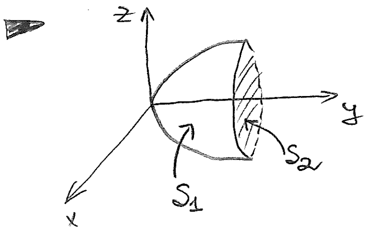
Hence: $$\iint_{S_3} \vec{F}\,d\vec{S} = \iint_{S_3} 3\,dS = 3\cdot A(S_3) = 3\cdot 5\cdot(2\pi\cdot 3) = 90\pi$$

Summarizing all the above, we see that

$$\iint_S \vec{F}\,d\vec{S} = 45\pi + 0 + 90\pi = \boxed{135\pi}$$

Remark: We on purpose illustrated two approaches:
- 1st Way is the most canonical
- 2nd Way is sometimes easier. (as we saw).

Ex 2: Let $\vec{F}(x,y,z) = \langle 0, y, -z\rangle$. Find the flux of $\vec{F}$ across the positively oriented S, which consists of the paraboloid $y = x^2+z^2$, $0 \le y \le 1$ and the disk $x^2+z^2 \le 1$, $y=1$.



This surface S consists of two parts: S_1 and S_2

- S_1 - part of the paraboloid $y = x^2+z^2$, $0\le y\le 1$
- S_2 - disk $x^2+z^2\le 1$, $y=1$.

So: $\iint_S \vec{F}\,d\vec{S} = \iint_{S_1} \vec{F}\,d\vec{S} + \iint_{S_2} \vec{F}\,d\vec{S}$

Flux across S_2

Parametrize S_2 via $\vec{r}(u,v) = \langle u\cos v, 1, u\sin v\rangle$, $\begin{matrix} 0\le u\le 1 \\ 0 \le v \le 2\pi\end{matrix}$

$$\left.\begin{matrix}\vec{r}_u = \langle \cos v, 0, \sin v\rangle \\ \vec{r}_v = \langle -u\sin v, 0, u\cos v\rangle\end{matrix}\right\} \Rightarrow \vec{r}_u \times \vec{r}_v = -u\cdot\vec{j}$$

But looking at the picture, it is clear that to get a vector pointing outwards, we need to take $-\vec{r}_u\times\vec{r}_v = u\cdot\vec{j}$.

Hence: $\iint_{S_2} \vec{F}\,d\vec{S} = \int_0^{2\pi}\int_0^1 \langle 0, 1, -u\sin v\rangle\cdot\langle 0, u, 0\rangle\,du\,dv = \boxed{\pi}$

Note: We could as in Ex1 immediately notice that $\vec{n} = \vec{j} \Rightarrow \vec{F}\cdot\vec{n} = 1$ on S_2

$\Rightarrow \iint_{S_2}\vec{F}\,d\vec{S} = A(S_2) = \pi$.

Flux across S_1

Parametrize S_1 via $\vec{r}(u,v) = \langle u, u^2+v^2, v\rangle$, (u,v) is subject to $u^2+v^2\le 1$.

$$\left.\begin{matrix}\vec{r}_u = \langle 1, 2u, 0\rangle \\ \vec{r}_v = \langle 0, 2v, 1\rangle\end{matrix}\right\} \Rightarrow \vec{r}_u\times\vec{r}_v = \langle 2u, -1, 2v\rangle.$$

To decide on $\pm\vec{r}_u\times\vec{r}_v$, pick $u=v=0 \Rightarrow \vec{r}_u\times\vec{r}_v = \langle 0,-1,0\rangle$ - points outwards $\Rightarrow$ we keep $\vec{r}_u\times\vec{r}_v$.

Hence: $\iint_{S_1}\vec{F}\,d\vec{S} = \iint_{u^2+v^2\le 1}\langle 0, u^2+v^2, -v\rangle\cdot\langle 2u, -1, 2v\rangle\,dA = \int_0^{2\pi}\int_0^1 (-r^2\cos^2\theta - 3r^2\sin^2\theta)\cdot r\,dr\,d\theta$

After straightforward computations (using $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$, $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$), we get $\iint_{S_1}\vec{F}\,d\vec{S} = -\pi$. Therefore: $\iint_S \vec{F}\,d\vec{S} = \pi - \pi = 0$ ∎

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* Today: Stokes' Theorem.

This is a 3D analogue of the Green's Theorem. It relates line integral over the boundary of a surface S and a flux across S.

Convention: The orientation of a surface S (given by a unit normal vector $\vec{n}$ at all points) induces the positive orientation of the boundary curve C. This means that if you walk in the positive direction around C with your head pointing in the direction of $\vec{n}$, then the surface is always on your left.

Stokes' Theorem: Let S be an oriented piecewise smooth surface bounded by a simple closed piecewise smooth curve C (endowed with a positive orientation). Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region that contains S. Then:

$$\boxed{\int_C \vec{F}\,d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\,dS}$$

We will use this result in two ways:

- reduce a computation of a line integral to a surface integral (which will often be easier to compute). Here we may choose any surface S whose boundary is the given curve C
- reduce a surface integral to a line integral, but this will require uncurling the original vector field

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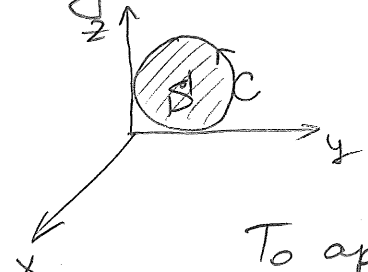
Ex3: Let C be a curve defined by the parametric equation

$C: x=0,\ y=2+2\cos t,\ z=2+2\sin t$ and t ranges from 0 to 2π.

Evaluate

$$\int_C x^2 e^{5z}\,dx + x\cos y\,dy + 3y\,dz.$$

First of all, let us note that we cannot compute this line integral in a straightforward way.



C - circle of radius 2, centered at $(0,2,2)$, in the yz-plane and it is oriented as shown.

To apply the Stokes' Theorem, we need to pick a surface S whose boundary is C. Simplest choice for S - the disk bounded by the circle!

$$\operatorname{curl}(x^2e^{5z}\vec{i} + x\cos y\cdot\vec{j} + 3y\cdot\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2e^{5z} & x\cos y & 3y \end{vmatrix} = 3\vec{i} + 5x^2e^{5z}\vec{j} + \cos y\cdot\vec{k}$$

So: $\int_C x^2e^{5z}dx + x\cos y\,dy + 3y\,dz = \iint_S \langle 3, 5x^2e^{5z}, \cos y\rangle\, dS$

At any point on S, $\vec{n} = \pm\vec{i}$ and to get an orientation compatible with that of C it is clear we must pick $\vec{n} = \vec{i} \Rightarrow$ get

$$\iint_S \langle 3, 5x^2e^{5z}, \cos y\rangle \cdot \langle 1,0,0\rangle\, dS = 3A(S) = 3\cdot\pi\cdot 2^2 = \boxed{12\pi}$$