Lecture #22

11/15/2018

* Last time: Stokes' Theorem

$$\boxed{\int_C F\,d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\,dS}$$

boundary of S with the "compatible" positive orientation

a surface with a chosen orientation

Remarks:

(1) We will apply this theorem both ways: more frequently to compute line integrals, but sometimes also to compute surface integrals

(2) Given C, you should choose any S whose boundary is C, and usually you shall take the simplest surface

(3) If you want to apply Stokes's Theorem to compute $\iint_S \vec{G}\,dS$, then this is possible iff $\operatorname{div}\vec{G}=0$ and in this case you uncurl $\vec{G}$, i.e. find $\vec{F}$ s.t. $\operatorname{curl}(\vec{F})=\vec{G}$, or replace S by another simpler surface with the same boundary.

(4) Given a surface S, its boundary is commonly denoted ∂S, so that

$$\int_{\partial S} \vec{F}\,d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\,dS$$

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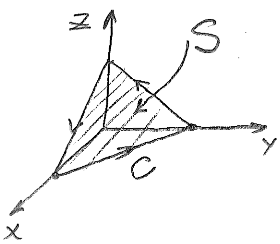
Ex1: Evaluate $\int_C \vec{F}\,d\vec{r}$, where $\vec{F} = \langle z^2, y^2, x\rangle$ and C is the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$ walked over in this direction/order.



- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = -\vec{j}\,(1-2z) = (2z-1)\vec{j}$
- Next, we need to choose a surface S whose boundary is C and the easiest choice is just to take the interior of the corresponding triangle.
- Next, we need to parametrize S, e.g. via

$$\vec{r}(u,v) = \langle u, v, \underbrace{1-u-v}_{f(u,v)}\rangle \quad , \quad \underbrace{(u,v) \text{ is s.t. } \begin{matrix} u\geq 0, v \geq 0 \\ u+v\leq 1\end{matrix}}_{\text{domain } D}$$

$\Downarrow$ general f-la

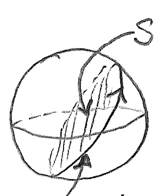
$$\vec{r}_u \times \vec{r}_v = \langle -f_u, -f_v, 1\rangle = \langle 1,1,1\rangle$$

- It is clear from the picture that if we choose this orientation of S, then the orientation of C given in the problem is positive!

$$\int_C \vec{F}\,d\vec{r} \overset{\text{Stokes}}{=} \iint_S \operatorname{curl}(\vec{F})\,dS = \iint_D \langle 0, \underbrace{2z-1}_{2(1-u-v)-1}, 0\rangle \cdot \langle 1,1,1\rangle\, dA = \int_0^1\int_0^{1-u} (1-2u-2v)\,dv\,du$$

$$= \int_0^1 \left[(1-2u)v - v^2\right]\Big|_{v=0}^{v=1-u} du = \int_0^1 \left((1-2u)(1-u) - (1-u)^2\right)du = \int_0^1 (u^2-u)\,du = \left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Big|_{u=0}^{u=1} = \boxed{-\frac{1}{6}}$$

Ex2: Evaluate $\int_C \vec{F}\,d\vec{r}$, where $\vec{F} = \langle z-y, -x-z, -x-y\rangle$ and C is the curve $x^2+y^2+z^2=4,\ y=z$ oriented counterclockwise when viewed from above.



C: intersection of a sphere and a plane

- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ z-y & -x-z & -x-y \end{vmatrix} = \vec{i}\,(-1+1) - \vec{j}\,(-1-1) + \vec{k}\cdot(-1+1) = 2\vec{j}$
- Next we choose a surface S whose boundary is C and the simplest choice is to take the enclosed disk S

Thus: $\int_C \vec{F}\,d\vec{r} = \iint_S 2\vec{j}\,dS$

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To compute $\iint_S 2\vec{j}\, d\vec{S}$, we can proceed 2 ways:

Cheap solution (not always applicable)

As S lies in the plane $y+z=0 \Rightarrow$ unit normal vector at every point is $\pm\frac{1}{\sqrt{2}}(0,1,-1)$ and looking at the picture it is clear that $\vec{n}=(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

But then $2\vec{j}\cdot\vec{n}=-\sqrt{2} \Rightarrow \iint_S 2\vec{j}\, d\vec{S} = -\sqrt{2}\cdot \underbrace{A(S)}_{\text{surface area of } S} = -\sqrt{2}\cdot\pi\cdot 2^2 = -4\sqrt{2}\pi$

Honest/Canonical solution

Parametrize S via $\vec{r}(u,v)=\langle u,v,v\rangle$, $(u,v)\in D=\{(u,v) \mid u^2+2v^2\le 4\}$.

$\Downarrow$

$\vec{r}_u\times\vec{r}_v=\langle 0,-1,1\rangle$ – and we take this (not $-\vec{r}_u\times\vec{r}_v$!)

Hence: $\iint_S 2\vec{j}\, d\vec{S} = \iint_D \langle 0,2,0\rangle\cdot\langle 0,-1,1\rangle\, dA = \iint_D -2\, dA = -2\, \mathrm{Area}(D)$

D is an ellipse and as we computed in the class $\mathrm{Area}(D)=\pi\cdot 2\cdot\sqrt{2}$

$\Rightarrow \int_C \vec{F}\, d\vec{r} = \boxed{-4\sqrt{2}\pi}$ ∎

Ex 3: Evaluate $\iint_S \mathrm{curl}(\vec{F})\, d\vec{S}$, where $\vec{F}=z^2\vec{i}-3xy\vec{j}+x^3y^3\vec{k}$ and S is the part of paraboloid $z=5-x^2-y^2$ above the plane $z=1$, oriented upwards.

Here we apply Stokes Thm other way around: $\iint_S \mathrm{curl}(\vec{F})\, d\vec{S} = \int_C \vec{F}\, d\vec{r}$.

The boundary C of S is the circle of radius 2 in the plane $z=1$, which is oriented counterclockwise when viewed from above.

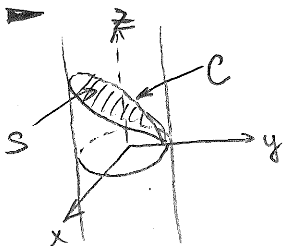
Parametrize C via $\vec{r}(t)=\langle 2\cos t, 2\sin t, 1\rangle$ with t ranging from 0 to 2π.

$\left.\begin{array}{l}\vec{F}(\vec{r}(t))=\langle 1, -12\cos t\sin t, 64\cos^3 t\sin^3 t\rangle \\ \vec{r}'(t)=\langle -2\sin t, 2\cos t, 0\rangle\end{array}\right\} \Rightarrow \vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = -2\sin t - 24\cos^2 t\sin t$

Hence: $\int_C \vec{F}\, d\vec{r} = \int_0^{2\pi}(-2\sin t-24\cos^2 t\sin t)\, dt = -24\int_0^{2\pi}\cos^2 t\sin t\, dt \overset{u=\cos t}{=\!=} 24\int_1^1 u^2\, du = \boxed{0}$ ∎

! Alternatively you could reduce to integrating over a disk enclosed by C

Ex 4: Evaluate $\int_C \vec{F}\, d\vec{r}$, where $\vec{F} = \langle y^2, -x, -z^2 \rangle$ and C is the curve of intersection of the plane $2y+z=1$ and the cylinder $x^2+y^2=1$ oriented counterclockwise when viewed from above.



- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & -x & -z^2 \end{vmatrix} = (-2y-1)\vec{k}$

- Pick S to be the interior of this circle.
- Parametrize S via $\vec{r}(u,v) = \langle u, v, 1-2v \rangle$ with the domain D for (u,v): $D = \{(u,v) \mid u^2+v^2 \le 1\}$
- $\vec{r}_u \times \vec{r}_v = \langle -\frac{\partial}{\partial u}(1-2v), -\frac{\partial}{\partial v}(1-2v), 1 \rangle = \langle 0, 2, 1 \rangle$
- Looking at the picture it's clear we pick $\vec{r}_u \times \vec{r}_v$ (not $-\vec{r}_u \times \vec{r}_v$)

So: $\int_C \vec{F}\, d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\, d\vec{S} = \iint_S \langle 0,0,-2y-1 \rangle\, d\vec{S} = \iint_D \langle 0,0,-2v-1 \rangle \cdot \langle 0,2,1 \rangle\, dA$

$= \iint_D (-2v-1)\, dA \underset{\substack{u = r\cos\theta \\ v = r\sin\theta}}{=} \int_0^{2\pi} \int_0^1 (-2r\sin\theta - 1)\cdot r\, dr\, d\theta = \int_0^{2\pi} \left(-\frac{2}{3}\sin\theta - \frac{1}{2}\right) d\theta = -\pi$ ■

Ex 5: Evaluate $\int_C \vec{F}\, d\vec{r}$, $\vec{F} = \langle y^2, 2x^2, e^{z^2} \rangle$ and C is the curve of intersection of the hemisphere $\begin{cases} x^2+y^2+z^2=25 \\ z \ge 0 \end{cases}$ and the cylinder $x^2+y^2=9$. oriented counterclockwise when viewed from above.

- C is the circle $\{(x,y,4) \mid x^2+y^2=9\}$ $\Rightarrow$ may pick S to be its interior parametrized via $\vec{r}(u,v) = \langle u, v, 4 \rangle$ with $D = \{(u,v) \mid u^2+v^2 \le 9\}$.
- $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & 2x^2 & e^{z^2} \end{vmatrix} = (4x-2y)\vec{k}$.

But: clearly $\vec{n} = \pm\vec{k}$ on S $\Rightarrow$ need to choose $\vec{n} = \vec{k}$ $\Rightarrow$ $\operatorname{curl}(\vec{F}) \cdot \vec{n} = 4x-2y$

$\Rightarrow \int_C \vec{F}\, d\vec{r} = \iint_S (4x-2y)\, dS = \iint_D (4u-2v)\, du\, dv = \int_0^{2\pi} \int_0^3 (4r\cos\theta - 4r\sin\theta) r\, dr\, d\theta = 0$ ■

Ex 6: Evaluate the integral $\iint_S \vec{G}\, dS$, where $\vec{G} = \langle e^{y^2}, e^{x^4}\cos(z^3), y^2x^2\rangle$ and S is the part of the paraboloid $z^2 = x^2+y^2$ below $z=4$, oriented upwards.

If we want to solve using Stokes' Theorem, we would need to find $\vec{F}$ such that $\mathrm{curl}(\vec{F}) = \vec{G}$.

However, due to the nature of the explicit formula for $\vec{G}$, it is clear that we don't have any chances to uncurl it.

Nevertheless, if we may show that such $\vec{F}$ exists, then we may replace S by any other surface with the same boundary.

- $\mathrm{div}\, \vec{G} = 0 \Rightarrow \vec{F}$ does exist
- Let S' be the disk, bounded by the boundary of S parametrized by $\vec{r}(u,v) = \langle u, v, 4\rangle$, $D = \{(u,v) \mid u^2+v^2 \le 4\}$.
- Moreover, the choice of the unit normal vector on S' is $\vec{k}$.

So: $\iint_S \vec{G}\, dS = \iint_{S'} \vec{G}\, dS = \iint_{S'} \langle e^{y^2}, e^{x^4}\cos(z^3), y^2x^2\rangle \cdot \langle 0,0,1\rangle\, dS = \iint_{S'} y^2x^2\, dS$

$$= \iint_D u^2v^2\, dA = \int_0^{2\pi}\int_0^2 r^4 \sin^2\theta \cos^2\theta \cdot r\, dr\, d\theta = \int_0^{2\pi} \frac{32}{6} \sin^2\theta\cos^2\theta\, d\theta =$$

$$= \int_0^{2\pi} \frac{32}{6} \cdot \frac{\sin^2(2\theta)}{4}\, d\theta = \int_0^{2\pi} \frac{32}{6\cdot 4} \cdot \frac{1-\cos(4\theta)}{2}\, d\theta = \boxed{\frac{4}{3}\pi}$$

! In the second extra problem on the homework you will be asked to actually uncurl a vector field, and then apply Stokes' Thm.